On morphological hierarchical representations for image processing and spatial data clustering

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Abstract Hierarchical data representations in the context of classification and data clustering were put forward during the fifties. Recently, hierarchical image representations have gained renewed interest for segmentation purposes. In this paper, we briefly survey fundamental results on hierarchical clustering and then detail recent paradigms developed for the hierarchical representation of images in the framework of mathematical morphology: constrained connectivity and ultrametric watersheds. Constrained connectivity can be viewed as a way to constrain an initial hierarchy in such a way that a set of desired constraints are satisfied. The framework of ultrametric watersheds provides a generic scheme for computing any hierarchical connected clustering, in particular when such a hierarchy is constrained.

Keywords image representation, segmentation, clustering, ultrametric, hierarchy, graphs, connected components, constrained connectivity, watersheds.

1 Introduction

A hierarchical representation of an image can be viewed as an ordered set or tree (acyclic graph) with some elementary components defining its leaves and the full image domain defining its root. Examples of elementary components are the regional minima/maxima/extrema, or the flat zones of the input image. This approach is interesting in all applications where the tree encoding the hierarchy offers a suitable basis for revealing structural information for filtering or segmentation purposes. Hierarchical representations predates developments in image processing and actually play a central role in classification by clustering methods (see for example [4] for an old but excellent review). In addition, hierarchical image segmentation can be seen as a hierarchical clustering of spatial data. For this reason, Sec. 2 briefly reviews fundamental concepts of classical hierarchical clustering methods (i.e, methods where the spatial location of the data points is usually not taken into account). Hierarchical image segmentation methods in a nutshell are presented in Sec. 3. Recent recent paradigms developed for the hierarchical representation of images in the framework of mathematical morphology known as constrained connectivity and ultrametric watersheds are then developed in Sec. 4 while highlighting their links with hierarchical clustering methods. The framework of ultrametric watersheds provides a generic scheme for computing any hierarchical connected clustering, in particular when such a hierarchy is constrained. Before concluding, the problem of transition pixels is briefly addressed in Sec. 5.

2 Hierarchical clustering

Because we are chiefly interested in image segmentation applications, we focus on clustering methods that are monothetic, partitional, and hierarchical. The term *hierarchical clustering* was first coined in [12]. A hierarchical clustering can be viewed as a sequence of nested clusterings such

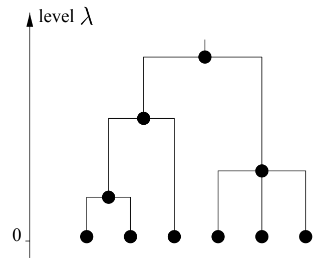


Figure 1: An example of dendrogram starting from 6 objects at the bottom of the hierarchy (level $\lambda = 0$). At the top of the hierarchy, there remains only one cluster containing all objects.

that a cluster at a given level is either identical to a cluster already existing at the previous level or is formed by unioning two or more clusters existing at the previous level. It is convenient to represent this hierarchy in the form of a tree called *dendrogram* [26].

By construction, a hierarchical clustering is parameterised by a non-negative real number λ indicating the level of a given clustering in the hierarchy. At the bottom level, this number is equal to zero and each object correspond to a cluster so that the finest possible partition is obtained. At the top level only one cluster containing all objects remains. Given any two objects, it is possible to determine the minimum level value for which these two objects belong to the same cluster. A key property of hierarchical clustering is that the function that measures this minimum level is an *ultrametric*. An ultrametric is a measurement that satisfies all properties of a metric (distance) plus a condition stronger than the triangle inequality and called ultrametric inequality. It states that the distance between two objects is lower than or equal to the maximum of the distances calculated from (i) the first object to an arbitrary third object and (ii) this third object to the second object. Denoting by d the ultrametric function and x, y, and z respectively the first, second and third objects, the ultrametric inequality corresponds to the following equation:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

The ultrametric property of hierarchical clustering was discovered simultaneously in [12, 11]. An example of dendrogram is displayed in Fig. 1.

The measure of similarity between the input objects requires the selection of a dissimilarity measurement. A dissimilarity measurement between the elements of a set X is a function $d^\star$ from $X \times X$ to the set of nonnegative real numbers satisfying the three following conditions: (i) $d^\star(x, y) \geq 0$ for all $x, y \in X$ (i.e., positiveness), (ii) $d^\star(x, x) = 0$ for all $x \in X$, and (iii) $d^\star(x, y) = d^\star(y, x)$ for all $x, y \in X$ (i.e., symmetry). Starting from an arbitrary dissimilarity measurement, it is possible to construct a hierarchical clustering with the ultrametric distance between any two objects (or clusters) being defined as the dissimilarity threshold level from which these two objects (or clusters) belong to the same cluster. In practice, this is achieved by an iterative procedure merging first the object pair with the smallest dissimilarity value so as to form a first non-trivial cluster (i.e., non reduced to one object). To proceed, the dissimilarity measurement between objects needs to be extended so as to be applicable to clusters. Let C_i and C_j denote two clusters obtained at a given iteration level. The dissimilarity between between these two clusters is naturally defined as a function of the dissimilarities between the objects belonging to these clusters:

$$d^\star(C_i, C_j) = f\{d^\star(x, y)) \mid x \in C_i \text{ and } y \in C_j\}.$$

Typical choices for the function f are the minimum or maximum. The maximum rule leads to the complete-linkage clustering (sometimes called maximum method) and dates back to [27]. Complete-linkage is subject to ties in case the current smallest dissimilarity value is shared by two or more clusters. On the other hand, the minimum rule is not subject to ties (and is therefore uniquely defined) and does not favour compact clusters. The resulting clustering is called the single-linkage clustering (sometimes called minimum method). Indeed, only the pair (link) with the smallest dissimilarity value is playing a role.

The single-linkage clustering is closely related to the minimum spanning tree computation [8]. While the single-linkage is not subject to ties, it is sensitive to the presence of objects of intermediate characteristics (transitions) that may occur between two clearly defined populations, see [28] for a detailed discussion as well as Sec. 5.

3 Hierarchical image segmentation

By analogy with hierarchical clustering, hierarchical segmentation can be defined as a family of fine to coarse image partitions (i.e., family of nested partitions) parameterised by a non-negative real number indicating the level of a given partition in the hierarchy. Hierarchical segmentation is useful to help the detection of objects in an image. In particular, it can be used to simplify the image in such a way that the elementary picture elements are not anymore the pixels but connected segments of pixels. Indeed, in image data, analogues to phonemes and characters correspond to structural primitives that compress the data to a manageable size without eliminating any possible final interpretations [2]. It should be emphasised that a hierarchical segmentation does not necessarily deliver segments directly corresponding to the searched objects. This happens for instance when an object is not characterised by some homogeneity/separation criteria but from the consideration of an a priori model of the whole object (e.g. perceptual grouping and Gestalt theory).

A brief survey of hierarchical image segmentation techniques is proposed in [23]. A recent review on hierarchical methods developed in mathematical morphology is presented in [16].

4 Constrained connectivity and ultrametric watersheds

4.1 Background definitions and notations

Following the notations of [7], we present some basic definitions to handle graphs.

We define a *graph* as a pair $X = (V, E)$ where V is a finite set and E is composed of unordered pairs of V, *i.e.*, E is a subset of $\{\{p, q\} \subseteq V \mid p \neq q\}$. Each element of V is called a *vertex or a point (of X)*, and each element of E is called an *edge (of X)*. If $V \neq \emptyset$, we say that X is *non-empty*.

As several graphs are considered in this paper, whenever this is necessary, we denote by $V(X)$ and by $E(X)$ the vertex and edge set of a graph X.

Let X be a graph. If $u = \{p, q\}$ is an edge of X, we say that p and q are *adjacent (for X)*. Let $\pi = \langle p_0, \ldots, p_\ell \rangle$ be an ordered sequence of vertices of X, π is *a path from p_0 to p_ℓ in X (or in V)* if for any $i \in [1, \ell]$, p_i is adjacent to p_{i-1}. In this case, we say that *p_0 and p_ℓ are linked for X*. We say that X *is connected* if any two vertices of X are linked for X.

Let X and Y be two graphs. If $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$, we say that *Y is a subgraph of X* and we write $Y \subseteq X$. We say that Y is a *connected component of X*, or simply a *component of X*, if Y is a connected subgraph of X which is maximal for this property, *i.e.*, for any connected graph Z, $Y \subseteq Z \subseteq X$ implies $Z = Y$.

Let X be a graph, and let $S \subseteq E(X)$. The *graph induced by S* is the graph whose edge set is S and whose vertex set is made of all points that belong to an edge in S, *i.e.*, $(\{p \in V(X) \mid \exists u \in S, p \in u\}, S)$.

Important remark. *Throughout this paper $G = (V, E)$ denotes a connected graph, and the letter V (*resp. *E) will always refer to the vertex set (*resp. *the edge set) of G. We will also assume that $E \neq \emptyset$. Let $S \subset E$. In the following, when no confusion may occur, the graph induced by S is also denoted by S.*

Typically, in applications to image segmentation, V is the set of picture elements (pixels) and E is any of the usual adjacency relations, *e.g.*, the 4- or 8-adjacency in 2D [13]. In all examples, 4-adjacency is used.

If $S \subset E$, we denote by $\overline{S}$ *the complementary set of S in E*, *i.e.*, $\overline{S} = E \setminus S$.

We consider in this paper weighted graphs, and either the points or the edges of a graph can be weighted. We denote the weight on the points of V by f, and the weights on the edges of E by F. For application to image processing, f is generally some information on the pixels (e.g., the grey level of the considered pixel), and F represents a dissimilarity (e.g., $F(\{p,q\}) = |f(x) - f(y)|$).

Two pixels p and q of an image f are α-connected if there exists a path going from p to q such that the dissimilarity between any two successive pixels of this path does not exceed the value of the local parameter α. By definition, a pixel is α-connected to itself. Accordingly, the α-connected component of a pixel p is defined as the set of image pixels that are α-connected to this pixel. We denote this connected component by α-CC(p): $\alpha\text{-CC}(p) = \{p\} \cup \big\{q \mid$ there exists a path $\mathcal{P} = \langle p = p_1, \ldots, p_n = q\rangle$, $n > 1$, such that $F(\{p_i, p_{i+1}\}) \le \alpha$ for all $1 \le i < n\big\}$.

4.2 Constrained connectivity

4.2.1 Definitions

The constrained connectivity paradigm [22, 23] originates from the need to develop a method preventing the formation of α-connected components whose range values exceed that specified by the local range parameter α. This is simply achieved by looking for the largest α-connected components satisfying a global range constraint referred to as the global range parameter denoted by ω:

$$(\alpha,\omega)\text{-CC}(p) = \bigvee \Big\{\alpha_i\text{-CC}(p) \;\Big|\; \alpha_i \le \alpha \text{ and } \mathsf{R}\Big(\alpha_i\text{-CC}(p)\Big) \le \omega\Big\},$$

where the range function R calculates the difference between the maximum and the minimum values of a nonempty set of intensity values. Note that the (α,ω)-connected components for $\alpha \ge \omega$ are equivalent to those obtained for $\alpha = \omega$. That is, when $\alpha \ge \omega$ the local range parameter does not play a role. This leads to the concept of (ω)-connected component[1]:

$$(\omega)\text{-CC}(p) = (\alpha \ge \omega, \omega)\text{-CC}(p) = \bigvee \Big\{\alpha_i\text{-CC}(p) \mid \mathsf{R}\Big(\alpha_i\text{-CC}(p)\Big) \le \omega\Big\}.$$

The corresponding global dissimilarity measurement $d^\star_\Omega$ between two pixels is defined by the smallest range of the α-connected components containing these two pixels. This dissimilarity measurement satisfies also the ultrametric inequality. Accordingly, we obtain the following equivalent definition of a (ω)-connected component: $(\omega)\text{-CC}(p) = \{q \mid d^\star_\Omega(p,q) \le \omega\}$. In contrast to what happens with the local dissimilarity measurement $d^\star_A$, the range of the values of arbitrary pairs of pixels belonging to the same (ω)-connected component is limited, the maximal value of this range being equal to ω. Therefore, the resulting clustering bears some resemblance to the complete linkage clustering suggested by Sørensen [27] but, contrary to the latter procedure, it is unequivocal. The generalisation of the concept of constrained connectivity to arbitrary constraints is presented in [22].

4.2.2 Component tree

Let (V, E, F) be the edge-weighted graph obtained from f by setting, for any $\{p,q\} \in E$, $F(\{p,q\}) = |f(p) - f(q)|$. It is easy to see that the connected components of the graph $(V, \{\{p,q\} \in E \mid F(\{p,q\}) \le \alpha\})$ are the α-connected components of f.

We denote by $F[\alpha]$ the graph $(V, \{\{p,q\} \in E \mid F(\{p,q\}) \le \alpha\})$. It is easy to check whether or not a given component C of $F[\alpha]$ satisfies $R(C) \le \omega$. Thus, for any p, (α,ω)-CC(p) is given by the largest C for which there exists $\lambda \le \alpha$ such that C is a component of $F[\lambda]$ containing p and satisfying the constraint $R(C) \le \omega$.

We define $\mathcal{C}(F)$ as the set composed of all the pairs $[\lambda, C]$, where $\lambda \in \mathbb{R}^+_0$ and C is a component of the graph $F[\lambda]$. We call *altitude of* $[\lambda, C]$ the number λ. We note that one can reconstruct F from $\mathcal{C}(F)$; more precisely, we have: $F(v) = \min\{\lambda \mid [\lambda, C] \in \mathcal{C}(F), v \in E(C)\}$. For any component C

[1] The parenthesis is not dropped to avoid confusion with α-connected components when the Greek letters are replaced by a numerical value indicating the actual value of the corresponding range parameter.

of F, we set $h(C) = \min\{\lambda \mid [\lambda, C] \in \mathcal{C}(F)\}$. We define $\mathcal{C}^\star(F)$ as the set composed by all $[h(C), C]$ where C is a component of F. The set $\mathcal{C}^\star(F)$, called the component tree of F [21, 20], is a finite subset of $\mathcal{C}(F)$ that is widely used in practice for image filtering (although it is generally defined for node-weighted graps). Note that the previous equation also holds for $\mathcal{C}^\star(F)$:

$$F(v) = \min\{\lambda \mid [\lambda, C] \in \mathcal{C}^\star(F), v \in E(C)\}.$$

Two components of $\mathcal{C}^\star(F)$ are either nested or disjoint, thus $\mathcal{C}^\star(F)$ is equivalent to a hierarchy. As any component $[\alpha, C]$ of $\mathcal{C}^\star(F)$ is an α-connected component, more precisely as $V(C)$ is an α-connected component, $\mathcal{C}^\star(F)$ is the hierarchy of α-connected component of F. Indeed, it is easy to see that drawing $\mathcal{C}^\star(F)$ amounts to drawing the dendrogram of the hierarchy. We can check whether or not it is an (α, ω)-connected component by verifying if $R(V(C)) \leq \omega$. In morphological terms, R can be seen as a flooding on $\mathcal{C}^\star(F)$.

4.2.3 Minimum spanning tree

As constrained connectivity yields a hierarchy, there is an underlying minimum spanning tree associated to it.

To any edge-weighted graph $X = (V(X), E(X), F)$, the number $F(X) = \sum_{u \in E(X)} F(u)$ is the *weight* of the graph. A connected graph spanning for V with minimum weight is a tree called a *minimum spanning tree* of F.

Points of the connected components of F are the same as the points of the connected components of any minimum spanning tree of F. More precisely, we have the following property.

Property 4.1 *Let X be a spanning tree of G. We write F_X for the restriction of F to the edges of X. The graph X is a minimum spanning tree of F if and only if:*

- *for any component of $[\alpha, C]$ of $\mathcal{C}^\star(F)$, there exists a component $[\alpha, C']$ of $\mathcal{C}^\star(F_X)$ such that $V(C) = V(C')$; and*
- *for any component of $[\alpha, C]$ of $\mathcal{C}^\star(F_X)$, there exists a component $[\alpha, C']$ of $\mathcal{C}^\star(F)$ such that $V(C) = V(C')$.*

Thus we can use F_X instead of F for doing some computation, which is less memory consuming as there are less edges in X than in G. Nevertheless, not all computation can be done on F_X.

4.2.4 Subdominant ultrametric

An ultrametric d, referred to as a *subdominant ultrametric*, is associated with the connected components obtained by successive and increasing thresholdings of F. For any two points p and q, $d(p, q)$ is given by the lowest altitude α of a component $[\alpha, C]$ of $\mathcal{C}^\star(F)$ such that p and q both belong to $V(C)$. Such α can be computed in constant time on $\mathcal{C}^\star(F)$, which itself can be computed in quasi-linear time [20].

Property 4.1 has the consequence that the subdominant ultrametric d of a graph is equal to the subdominant ultrametric of any of its minimum spanning trees X; in particular, $d(p, q)$ is equal to the highest value of F along the unique elementary path linking p to q in X.

4.3 Ultrametric watersheds : a unifying framework for hierarchical segmentation

We have several different ways to deal with hierarchies: dendrograms and minimum spanning trees. In the case where a hierarchy is made of connected regions, then we can also use its component tree. None of this three tools allows for an easy visualisation of a given hierarchy as an image. We now introduce ultrametric watershed [17, 19] as a tool that helps visualising a hierarchy: we stack the contours of the regions of the hierarchy; thus, the more a contour of a region is present in the hierarchy, the more visible it is. Ultrametric watershed is the formalisation and the caracterisation of a notion introduced under the name of *saliency map* [18].

4.3.1 Ultrametric watersheds

The formal definition of ultrametric watershed relies on the topological watershed framework [3].

Let X be a graph. An edge $u \in \overline{E(X)}$ is said to be *W-simple (for X)* if X has the same number of connected components as $X + u = (V(X), E(X) \cup \{u\})$.

An edge u such that $F(u) = \lambda$ is said to be *W-destructible (for F) with lowest value* λ_0 if there exists λ_0 such that, for all λ_1, $\lambda_0 < \lambda_1 \leq \lambda$, u is W-simple for $F[\lambda_1]$ and if u is not W-simple for $F[\lambda_0]$.

A *topological watershed (on G)* is a map that contains no W-destructible edges.

An *ultrametric watershed* is a topological watershed F such that $F(v) = 0$ for any v belonging to a minimum of F.

There exists a bijection between ultrametric distances and hierarchies of partitions [12]; in other word, to any hierarchy of partitions is associated an ultrametric, and conversely, any ultrametric yields a hierarchy of partitions, see also Sec. 2. Similarly, there exists a bijection between the set of hierarchies of *connected partitions* and the set of ultrametric watersheds [17, 19].

4.3.2 Usage: gradient and dissimilarity

Constrained connectivity is a hierarchy of flat zones of f, in the sense where the 0-connected components of f are the zones of f where the intensity of f does not change. In a continuous world, such zones would be the ones where the gradient is null, *i.e.* $\nabla f = 0$. However, the space we are working with is discrete, and a flat zone of f can consist in a single point. In general, it is not possible to compute a gradient on the points or on the edges such that this gradient is null on the flat zones. To compute a gradient on the edges such that the gradient is null on the flat zones, we need to "double" the graph, for example we can do that by doubling the number of points of V and adding one edge between each new point and the old one.

More precisely, if we denote the points of V by $V = \{p_0, \ldots, p_n\}$, we set $V' = \{p'_0, \ldots, p'_n\}$ (with $V \cap V' = \emptyset$), and $E' = \{\{p_i, p'_i\} \mid 0 \leq i \leq n\}$. We then set $V_1 = V \cup V'$ and $E_1 = E \cup E'$.

By construction, as $G = (V, E)$ is a connected graph, the graph $G_1 = (V_1, E_1)$ is a connected graph. We also extend f to V', by setting, for any $p' \in V'$, $f(p') = f(p)$, where $\{p, p'\} \in E'$.

We set, as in section 4.2.2, $F(\{p, q\}) = |f(p) - f(q)|$. The map F can be seen as the "natural gradient" of f [15]. We can then apply the same scheme on this F as in section 4.2.2 to find the hierarchy of constrained connectivity.

The main difference with section 4.2.2 lies in the fact that $F[\alpha]$ is induced by $\{\{p, q\} \in E_1 \mid F(\{p, q\}) \leq \alpha\}$, i.e. there is no isolated pixel in the hierarchy. In particular, the minima of F are the connected components of $\{\{p, q\} \in E_1 \mid F(\{p, q\}) = 0\}$, i.e., they are the 0-connected components of f, which was not the case before.

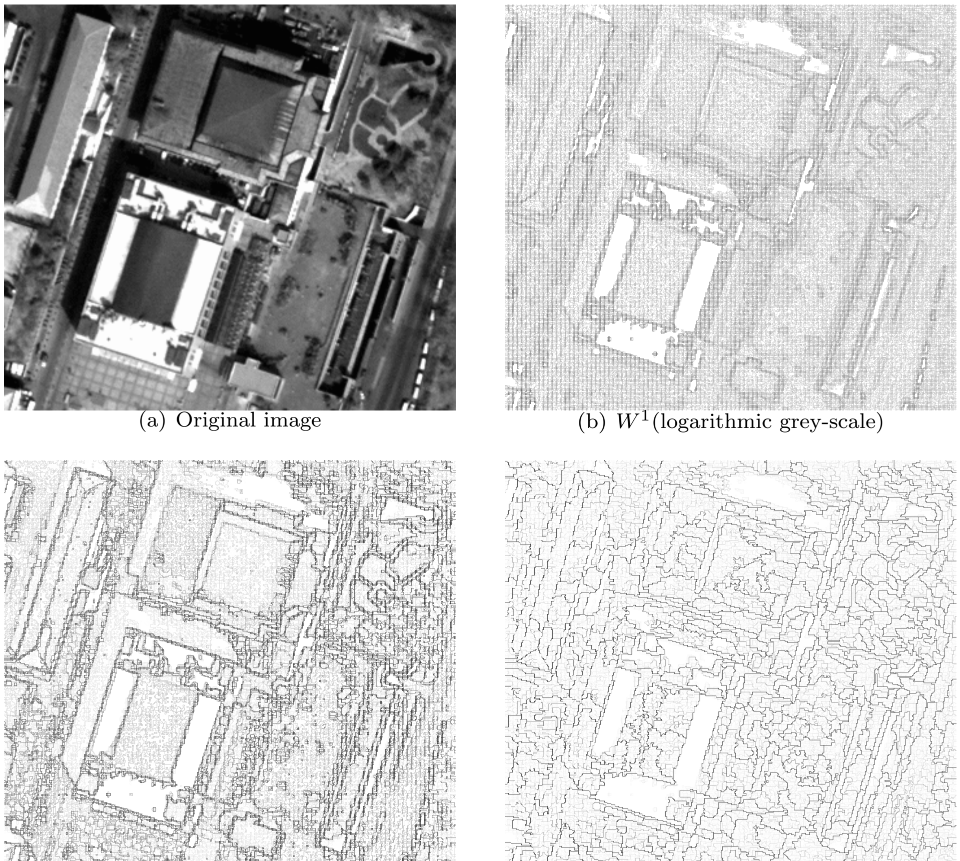
As in Sec. 4.2.2, finding the (α, ω)-CC can be done by filtering the ultrametric watershed W of F with R that acts as a flooding on the toplogical/ultrametric watershed W of F, and then finding a (topological) watershed of the filtered image. Repeating these steps, we build the constrained connectivity hierarchy. In effect, we are viewing a hierarchy as an image (edge-weighted graph) and transforming it into another hierarchy/image.

Thus, classical tools from mathematical morphology can be applied to constrain any hierarchy. Similar examples exist in the literature, for example [9], where the authors compute what they called a non-horizontal cut in the hierarchy, in other words, they compute a flooding on a watershed. In their framework, the flooding is controlled by an energy.

The advantages of using an ultrametric watershed are numerous. Let us mention the two following ones:

1. an ultrametric watershed is visible; a dendrogram or a component tree can be drawn, but less information is available from such a drawing, and visualising an MST is not really useful;

2. an ultrametric watershed allows to use any information in the contours between regions; such information is not available on the component tree, and is only partially available with an MST (which contains only the pass between regions).



(a) Original image

(b) W^1(logarithmic grey-scale)

(c) W^2

(d) Area-filtering ultrametric watershed

Figure 2: Soille's (α, ω)-constrained connectivity hierarchy. (a) Original image (extract from the panchromatic channel of a Quickbird Imagery © DigitalGlobe Inc., 2007, distributed by Eurimage). (b) Ultrametric watershed W^1 for the α-connectivity. (c) Ultrametric watershed W^2 for the constrained connectivity. (d) Ultrametric watersheds corresponding to one of the possible hierarchies of area-filterings on W^2.

Let us note that, for all application usages, those concepts are equivalent: even their respective computational time is nearly identical; thus we can choose the one the most adapted to the desired usage.

Visualising the hierarchy of constrained connectivity as an ultrametric watershed allows to assess some of its qualities. One can notice in Fig. 2.c a large number of transition regions (small undesirable regions that persist in the hierarchy), which is the topic of the next section.

5 Transition pixels

Constrained connectivity prevents the formation of connected components that would otherwise be created in case samples of intermediate value (transition pixels) between two populations (homogeneous image structures) are present. Indeed, these components would violate the global range or other appropriate constraint. However, sometimes the formation of two distinct connected components cannot occur at all. In the extreme case represented in Fig. 3. either each pixel is a connected component (flat zone) or there is a unique connected component. One way to address this problem is to propose a definition of transition pixels and perform some pre-processing to

0	1	0	7	8	7	8
1	0	1	6	7	8	7
0	1	0	5	8	7	8
1	0	1	4	7	8	7
0	1	0	3	8	7	8
1	0	1	2	7	8	7
0	1	0	1	8	7	8

0	1	0	7	8	7	8
1	0	1	6	7	8	7
0	1	0	5	8	7	8
1	0	1	4	7	8	7
0	1	0	3	8	7	8
1	0	1	2	7	8	7
0	1	0	1	8	7	8

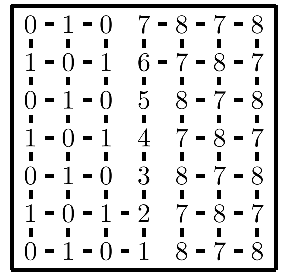


Figure 3: A synthetic sample image with its intensity values and its two possible partitions into constrained connected components *whatever* the considered constraints in case standard α-connectivity is used in the definitions. The two homogeneous regions show intensity variations of 1 level while the ramp between the two regions also proceeds by steps of 1 intensity level. In the image at the right, adjacent pixels are linked by an edge if and only if their range does not exceed 1.

suppress them. This approach is advocated in [25, 24]. For example, assuming that *local* extrema correspond to non-transition pixels, they are extracted on then considered as seeds whose values are propagated in the input image using a seeded region growing algorithm [1]. Note that this approach is linked with contrast enhancement techniques since it aims at increasing the external isolation of the obtained connected components. A number of classical morphological schemes (*e.g.*, area filtering of the ultrametric watershed) can be used to remove those transition zones (see Fig. 2.d for an example).

Another approach is to substitute the α-connectivity with a more restrictive connectivity. Indeed, the local range parameter α defined in [23] as the intensity difference between adjacent pixels can be viewed as a special case of dissimilarity measurement. Although this measurement is the most natural, other dissimilarity measurements may be considered. For example, the following alternative definition of alpha-connectivity may be considered to tackle the problem of transition regions. Let the α-degree of a pixel (node) be defined as the number of its adjacent pixels that are within a range equal to α:

$$\alpha\text{-deg}(p) = \#\{q \mid \{p, q\} \in E \text{ and } |f(q) - f(p)| \leq \alpha\}.$$

Then two pixels p and q are said to be α_n-connected if and only if there exists an α-path connecting them such that every pixel of the path has a α-degree greater of equal to n. We obtain therefore the following definition for the α_n-connected component of a pixel p:

$$\begin{aligned}\alpha_n\text{-CC}(p) \quad = \quad & \{p\} \cup \{q \mid \text{ there exists a path } \langle p = p_1, \ldots, p_n = q \rangle,\, n > 1, \\ & \text{such that } |f(p_i) - f(p_{i+1}| \leq \alpha \text{ and } \alpha\text{-deg}(p_i) \geq n\}.\end{aligned}$$

If necessary, other constraints can be considered. Note that α-connectivity is a special case of α_n-connectivity obtained for $n = 1$. In addition, the following nesting property holds:

$$\alpha_{n'}\text{-CC}(p) \subseteq \alpha_n\text{-CC}(p),$$

where $n \leq n'$. α_n-connectivity satisfies all properties of an equivalence relation and therefore also partitions the image definition domain into unique maximal connected components. An example is provided in Fig. 4. In this example, the non singleton 1_3-connected components match the core of the two homogeneous regions. Singleton connected components correspond to pixels whose degree is smaller than 3. Non-singleton connected components can be used as seeds for coarsening the obtained partition. Special care is needed to produce connected components matching one-pixel thick non-transition regions and will be discussed in an extended version of this paper.

0	1	0	7	8	7	8
1	0	1	6	7	8	7
0	1	0	5	8	7	8
1	0	1	4	7	8	7
0	1	0	3	8	7	8
1	0	1	2	7	8	7
0	1	0	1	8	7	8

2	3	2	2	3	3	2
3	4	3	3	4	4	3
3	4	3	2	3	4	3
3	4	3	2	3	4	3
3	4	3	2	3	4	3
3	4	4	3	3	4	3
2	3	3	2	2	3	2

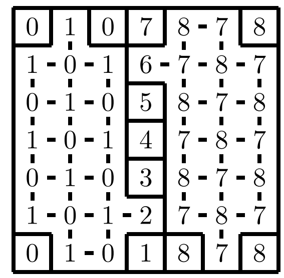


Figure 4: A synthetic sample image with its intensity values, the corresponding 1-deg map, and 1_3-connected components.

6 Conclusion and perspectives

In this paper, we have presented several equivalent tools dealing with hierarchies of connected partitions. Such a review invites us to look more closely at links between what have been done in different research domains as, for example, between clustering and lattice theory [10]. A first step in that direction is [5], and there is a need for in-depth study of operators acting on lattices of graphs [6]. The question of transition pixels is not only a theoretical one, regarding its significance for applications. Finally, we want to stress the importance of having framework allowing a generic implementation of existing algorithms, not limited to the pixel framework, but also able to deal transparently with edges, or, more generally, with graphs and complexes [14].

References

[1] R. Adams and L. Bischof. Seeded region growing. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 16(6):641–647, 1994. doi: 10.1109/34.295913.

[2] N. Ahuja. On detection and representation of multiscale low-level image structure. *ACM Computing Surveys*, 27(3):304–306, September 1995. doi: 10.1145/212094.212099.

[3] Gilles Bertrand. On topological watersheds. *J. Math. Imaging Vis.*, 22(2-3):217–230, 2005. ISSN 0924-9907. doi: http://dx.doi.org/10.1007/s10851-005-4891-5.

[4] R. Cormack. A review of classification (with discussion). *Journal of the Royal Statistical Society A*, 134:321–367, 1971.

[5] J. Cousty, L. Najman, and J. Serra. Raising in watershed lattices. In *15th IEEE ICIP'08*, pages 2196–2199, San Diego, USA, October 2008.

[6] Jean Cousty, Laurent Najman, and Jean Serra. Some morphological operators in graph spaces. In *ISMM 2009*, LNCS.

[7] R. Diestel. *Graph Theory.* Graduate Texts in Mathematics. Springer, 1997.

[8] J. Gower and G. Ross. Minimum spanning trees and single linkage cluster analysis. *Applied Statistics*, 18(1):54–64, 1969. URL `http://www.jstor.org/stable/2346439`.

[9] L. Guigues, J.-P. Cocquerez, and H. Le Men. Scale-sets image analysis. *International Journal of Computer Vision*, 68(3):289–317, 2006.

[10] L. Hubert. Some extension of Johnson's hierarchical clustering. *Psychometrika*, 37:261–274, 1972. doi: 10.1007/BF02306783.

[11] C. Jardine, N. Jardine, and R. Sibson. The structure and construction of taxonomic hierarchies. *Mathematical Biosciences*, 1(2):173–179, 1967. doi: 10.1016/0025-5564(67)90032-6.

[12] S. Johnson. Hierarchical clustering schemes. *Psychometrika*, 32(3):241–254, September 1967.

[13] T.Y. Kong and A. Rosenfeld. Digital topology: Introduction and survey. *Comput. Vision Graph. Image Process.*, 48(3):357–393, 1989.

[14] Rolland Levillain, Thierry Géraud, and Laurent Najman. Writing Reusable Digital Geometry Algorithms in a Generic Image Processing Framework. In *Workshop on Applications of Digital Geometry and Mathematical Morphology*, 2010. these proceedings.

[15] Claudio Mattiussi. The Finite Volume, Finite Difference, and Finite Elements Methods as Numerical Methods for Physical Field Problems. *Advances in Imaging and Electron Physics*, 113:1–146, 2000. doi: NA.

[16] Fernand Meyer and Laurent Najman. Segmentation, minimum spanning tree and hierarchies. In Laurent Najman and Hugues Talbot, editors, *Mathematical morphology: from theory to applications*, chapter 9, pages 255–287. Wiley-ISTE, 2010.

[17] L. Najman. Ultrametric watersheds. In M.H.F. Wilkinson and J.B.T.M. Roerdink, editors, *Proc. ISMM 2009*, volume 5720 of *Lecture Notes in Computer Science*, pages 181–192, 2009.

[18] L. Najman and M. Schmitt. Geodesic saliency of watershed contours and hierarchical segmentation. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 18(12):1163–1173, 1996.

[19] Laurent Najman. Ultrametric watersheds: a bijection theorem for hierarchical edge-segmentation. *CoRR*, abs/1002.1887, 2010.

[20] Laurent Najman and Michel Couprie. Building the component tree in quasi-linear time. *IEEE TIP*, 15(11):3531–3539, 2006.

[21] P. Salembier, A. Oliveras, and L. Garrido. Anti-extensive connected operators for image and sequence processing. *IEEE TIP*, 7(4):555–570, April 1998.

[22] P. Soille. On genuine connectivity relations based on logical predicates. In *Proc. of 14th Int. Conf. on Image Analysis and Processing, Modena, Italy*, pages 487–492. IEEE Computer Society Press, September 2007. doi: 10.1109/ICIAP.2007.4362825.

[23] P. Soille. Constrained connectivity for hierarchical image partitioning and simplification. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 30(7):1132–1145, July 2008. doi: 10.1109/TPAMI.2007.70817.

[24] P. Soille. Constrained connectivity for the processing of multispectral images. *International Journal of Remote Sensing*, 2010.

[25] P. Soille and J. Grazzini. Constrained connectivity and transition regions. *Lecture Notes in Computer Science*, 5720:59–69, 2009. doi: 10.1007/978-3-642-03613-2_6.

[26] R. Sokal and P. Sneath. *Principles of Numerical Taxonomy.* W. H. Freeman and Company, San Fransisco and London, 1963.

[27] T. Sørensen. A method of establishing groups of equal amplitude in plant sociology based on similarity of species content and its applications to analyses of the vegetation of Danish commons. *Biologiske Skrifter*, 5(4):1–34, 1948.

[28] D. Wishart. A generalization of nearest neighor which reduces chaining effect. In A. Cole, editor, *Numerical Taxonomy*, pages 282–311. Academic Press, New York, 1969.