Master 2 "SIS"

Digital Geometry

Topic 4: Discrete lines

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October 24, 2012

Straight line

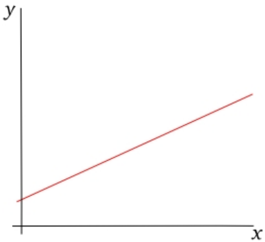
Definition (Straight line)

A line in the Euclidean space $\mathbb{R}^2$ *is defined by*

$$\mathbf{L} = \{(x, y) \in \mathbb{R}^2 : \alpha x + \beta y + \gamma = 0\}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

In general, we have a normalization such that $|\alpha| + |\beta| = 1$, $\alpha^2 + \beta^2 = 1$.



Discretization of straight line

Definition (Discrete line)

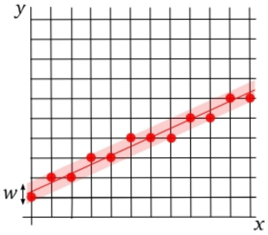
The discrete line of $\mathbf{L}$ *in* $\mathbb{Z}^2$ *is defined by*

$$D(\mathbf{L}) = \{(p,q) \in \mathbb{Z}^2 : 0 \leq \alpha p + \beta q + \gamma' \leq \omega\}$$

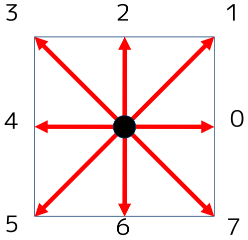
where ω *is called the* thickness.

The values of γ' and ω depend on the model of discretization.

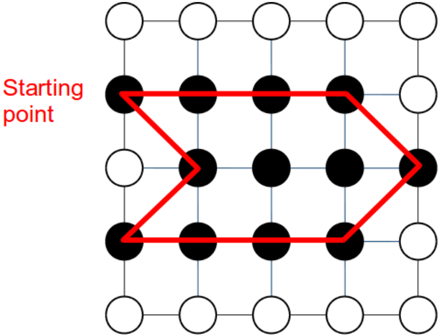
- **Grid-intersection**: $\gamma' = \gamma + \frac{\max(|\alpha|,|\beta|)}{2}$, $\omega = \max(|\alpha|,|\beta|)$.
- **Super-cover (outer Jordan)**: $\gamma' = \gamma + \frac{|\alpha|+|\beta|+1}{2}$, $\omega = |\alpha| + |\beta| + 1$.
- **Gauss** (half-plane): $\gamma' = \gamma$, ω decides the m-connectedness of the half-plane border: $\omega = \max(|\alpha|,|\beta|) - 1$ for $m = 8$.



Freeman code



Freeman code



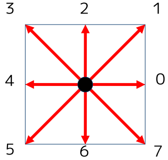
Chain code: 7500013444

Properties of discrete lines

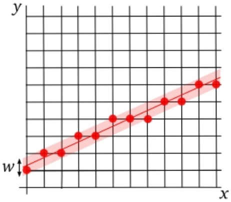
Criteria of Freeman (1974)

For discrete lines (by *grid-intersection* discretization), the **Freeman code** verify the following three properties:

1. the code contains **at most two different values**;
2. those two values differ **at most by one unit** (modulo 8) ;
3. one of the two values **appears isolatedly** and its appearances are uniformly spaced in the code.



Freeman code



Discrete line : 10101001010...

Properties of discrete lines (cont.)

Definition (Chord property (Rosenfeld, 1974))

A set of discrete points $\mathbf{X}$ *satisfies the* chord property *if for every pair of points* $\mathbf{p}$ *and* $\mathbf{q}$ *of* $\mathbf{X}$ *and for every point* $\mathbf{r} = (r_x, r_y)$ *of the real segment between* $\mathbf{p}$ *and* $\mathbf{q}$*, there exists a point* $\mathbf{s} = (s_x, s_y)$ *of* $\mathbf{X}$ *such that* $\max(|s_x - r_x|, |s_y - r_y|) < 1$.

- It proves that a discrete curve is a discrete line segment if and only if it owns the chord property.
- It allows to show the two first criteria of Freeman and to deduce a number of properties that specify the third criterion.
- There are a number of algorithms for recognizing a discrete straight line based on this property.

Bresenham line-drawing algorithm

Algorithm: drawing a discrete line (Bresenham, 1962)

Input: Two discrete points (x_1, y_1), (x_2, y_2) (s.t. $x_2 - x_1 \geq y_2 - y_1 > 0$)
Output: Line segment between the two points

- $d_x = x_2 - x_1$, $d_y = y_2 - y_1$;
- $y = y_1$; initialization
- $e = d_x$; value of initial error
- **for** x **from** x_1 **to** x_2 **do**
  - put the pixel (x, y);
  - $e = e + 2d_y$;
  - **if** $e \geq 2d_x$ **then**
    - $y = y + 1$;
    - $e = e - 2d_x$;

Can we consider **rounding** instead of truncation?

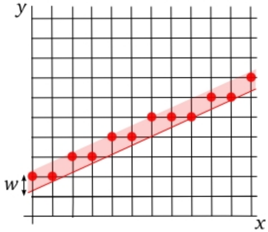
Arithmetic definition of discrete lines

Definition (Arithmetic discrete line)

A discrete line of parameters (a, b, c) and of arithmetic thickness w where $a, b, c \in \mathbb{Z}$ and $gcd(a, b) = 1$ is defined as

$$D(a, b, c, w) = \{(p, q) \in \mathbb{Z}^2 : 0 \leq ap + bq + c < w\}.$$

The thickness parameter w allows to control the connectedness of the line.

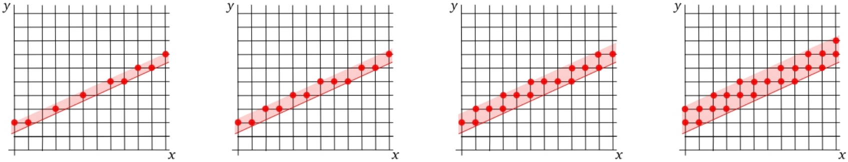


Thickness and connectedness of discrete lines

Theory

Let $D(a, b, c, w)$ be a discrete line, then:

1. *if $w < \max(|a|, |b|)$, it is not connected;*
2. *if $w = \max(|a|, |b|)$, it is a 8-curve ;* naive line
3. *if $\max(|a|, |b|) < w < |a| + |b|$, it is a $*$-curve (its two successive points are 4-neighboring or strictly 8-neighboring);*
4. *if $w = |a| + |b|$, it is a 4-curve;* standard line
5. *if $w > |a| + |b|$, it is said* thick.



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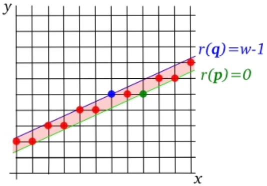
Remainder and leaning point of discrete lines

Definition (Remainder)

The remainder *associated to a point* $\mathbf{p} = (p_x, p_y)$ *of* $D(a, b, c, w)$ *is an integer value defined by*

$$R(\mathbf{p}) = ap_x + bp_y + c.$$



- When the remainder is 0, $\mathbf{p}$ is called a **lower leaning point**.
- When the remainder is $w - 1$, $\mathbf{p}$ is called a **upper leaning point**.

We can generalize the Bresenham algorithm by using the *remainder* instead of the *error e*.

Arithmetic line drawing algorithm

Algorithm: drawing an arithmetic (naive) line

Input: Two discrete points (x_1, y_1), (x_2, y_2) and c
Output: Line segment between the two points

- $b = x_2 - x_1$, $a = y_2 - y_1$;
- $y = y_1$;
- $r = ax_1 + by_1 + c$;
- **for** x **from** x_1 **to** x_2
  - put the pixel (x, y);
  - $r = r + a$;
  - **if** $r \geq b$ **then**
    - $y = y + 1$;
    - $r = r - b$;

We consider here that $x_2 - x_1 \geq y_2 - y_1 > 0$.

The value of c is initially chosen such that $0 \leq ax_1 + by_1 + c < b$.

Discrete line recognition

Problem (Discrete line recognition)

Given a discrete curve **X** *(more generally, a set of discrete points) do the points of* **X** *belong to a discrete line?*

Yes *or* **No**

If yes, what are the parameters of this discrete line?

There are many recognition algorithms with linear complexity.

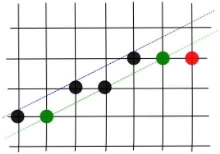
1. **approach of linear programming** (Werman et al, 1987): verify the existence of feasible (real) solutions.
2. **linguistic approach** (Smeulders, Dorst, 1991): use the linguistic properties of discrete lines.
3. **approach based on preimage** (Lindenbaum, Bruckstein, 1993): use the properties of discrete lines in the dual space, called preimages.
4. **arithmetic approach** (Debled-Rennesson, Reveillès, 1995): verify the existence of integer solutions by using arithmetic properties.

Incremental algorithm for arithmetic line recognition: initial situation

Let

- $\mathbf{S}$ be a segment of naive line $D(a, b, c)$ with $0 \leq a < b$,
- $\mathbf{q} = (x_{\mathbf{q}}, y_{\mathbf{q}})$ be the point of the greatest abscissa of $\mathbf{S}$,
- $\mathbf{l}$ and $\mathbf{l'}$ be the lower leaning points of minimum and maximum abscissas of $\mathbf{S}$,
- $\mathbf{u}$ and $\mathbf{u'}$ be the upper leaning points of minimum and maximum abscissas of $\mathbf{S}$.

By adding a point $\mathbf{p} = (x_{\mathbf{p}}, y_{\mathbf{p}})$ connected to $\mathbf{S}$ such that $x_{\mathbf{p}} = x_{\mathbf{q}} + 1$, we verify if $\mathbf{S}' = \mathbf{S} \cup \{\mathbf{p}\}$ is still a naive line segment.

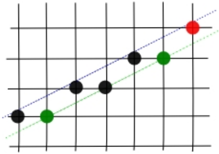


Incremental algorithm for arithmetic line recognition

Theory (Debled-Rennesson and Reveillès, 1995)

We have

1. *if* $0 < r(\mathbf{p}) < b$, $\mathbf{S}'$ *is a naive line segment* $D(a, b, c)$*;*
2. *if* $r(\mathbf{p}) < -1$ *or* $b < r(\mathbf{p})$, *then* $\mathbf{S}'$ *is not a naive line segment;*
3. *if* $r(\mathbf{p}) = -1$, *then* $\mathbf{S}'$ *is a naive line segment* $D(y_{\mathbf{p}} - y_{\mathbf{u}}, x_{\mathbf{p}} - x_{\mathbf{u}}, -ax_{\mathbf{p}} + by_{\mathbf{p}})$*;*
4. *if* $r(\mathbf{p}) = b$, *then* $\mathbf{S}'$ *is a naive line segment* $D(y_{\mathbf{p}} - y_{\mathbf{l}}, x_{\mathbf{p}} - x_{\mathbf{l}}, -ax_{\mathbf{p}} + by_{\mathbf{p}} + b - 1)$.



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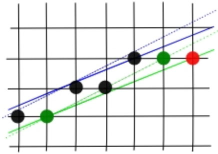

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Farey sequence

Definition (Farey sequence (Hardy and Write, 1979))

The Farey sequence of order n, F_n, is the sequence of irreducible fractions between 0 and 1, whose denominators are less than or equal to n, in ascending order.

If $0 <= h <= k <= n$ and $\gcd(h, k) = 1$, then $\frac{h}{k}$ is in F_n.

Example (F_5)

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

Structure of the Farey sequence: Stern-Brocot tree

Property (Neighborhood)

If $\frac{a}{b}$ and $\frac{c}{d}$ are neighboring in a Farey sequence, with $\frac{a}{b} < \frac{c}{d}$, then their difference is equal to $\frac{1}{bd}$.

Property (Median)

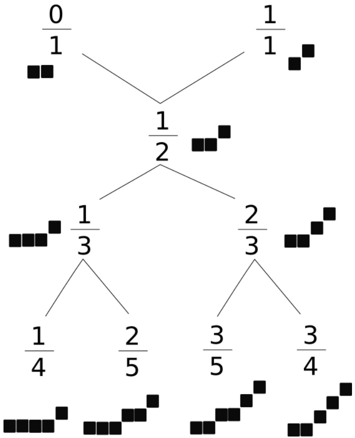
If $\frac{a}{b}$, $\frac{p}{q}$ and $\frac{c}{d}$ are neighboring in a Farey sequence such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$, then $\frac{p}{q}$ is the median of $\frac{a}{b}$ and $\frac{c}{d}$ such as

$$\frac{p}{q} = \frac{a+c}{b+d}.$$

These properties allow to construct the *Stern-Brocot tree*.

Stern-Brocot tree and discrete lines

Each vertex $\frac{h}{k}$ of the tree corresponds to a pattern (motif) associated to the discrete line of slope $\frac{h}{k}$.



Updating parameters of the incremental discrete line recognition algorithm indicates moving from the tree root to a leaf. (Debled-Rennesson, 1995)

Preimage of a discrete line segment

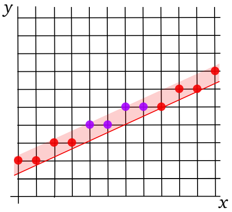
Definition (Preimage of a discrete line segment (Anderson, Kim, 1985))

Given a discrete line segment S, the preimage $\mathcal{P}(S)$ is defined as the set of all lines $y = sx + t$ having the same discretization S.

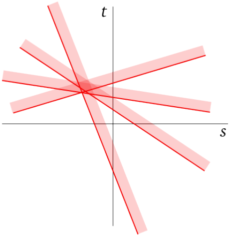
Each point $(p_i, q_i) \in S$, $i = 1, 2, \ldots, |S|$, satisfies

$$0 \leq sp_i + q_i + t < 1.$$

Thus, $\mathcal{P}(S)$ in the parameter space (s, t) is the intersection of $2|S|$ half-planes or $|S|$ strips of height 1.



Discrete line segment



Preimage

Properties of preimages and incremental discrete line recognition

Property (Preimage (Dorst, Smeulders, 1984))

Any preimage $\mathcal{P}(S)$ forms a convex polygon with at most four vertices in the parameter space. If $\mathcal{P}(S)$ has four vertices, then two of them have a common s-coordinate, which is between the s-coordinates of the other two vertices.

Property (Preimage (McIlroy, 1984))

The vertices of $\mathcal{P}(S)$ have s-coordinates that form a Farey sequence of order $\max(p_0, |S| - p_0)$.

These properties allow to improve the **incremental algorithm for discrete line recognition** with optimal complexity in linear time and constant space. For checking if the set $\mathbf{S}' = \mathbf{S} \cup \{(p, q)\}$ is a discrete line segment, we verify if

$$\mathcal{P}(S') = \mathcal{P}(S) \cap \{(s, t) | 0 \leq sp + q + t < 1\}$$

is not empty. (Lindenbaum, Bruckstein, 1993)

Applications of discrete line recognition

The discrete line recognition allow us to:

- study the *parallelism*, *colinearity*, *orthogonality*, *convexity* in the discrete space;
- estimate geometric properties of discrete object borders, such as the *length* of a curve, *tangent* and *curvature* at a point in a curve, etc.;
- make a segmentation of a discrete curve into line segments (*polygonal approximation*).

If there is noise in discrete object border, we need to modify the problem.

Discrete line fitting

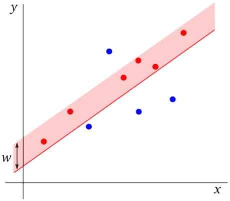
Problem (Discrete line fitting)

Given a finite set of discrete points such that

$$\mathbf{S} = \{\mathbf{p}_i \in \mathbb{Z}^2 \ : \ i = 1, 2, \ldots, N\},$$

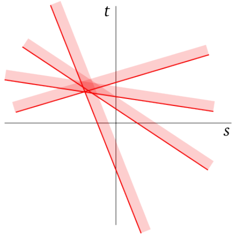
we seek the **maximum subset** *of* $\mathbf{S}$ *whose elements are contained by some discrete line* $\mathbf{D}(\mathbf{L})$.

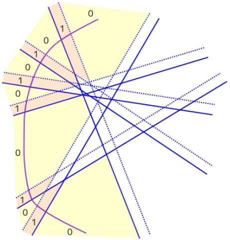
Points $\mathbf{p}_i \in \mathbf{S}$ are called **inliers** if $\mathbf{p}_i \in \mathbf{S} \cap \mathbf{D}(\mathbf{L})$; otherwise, they are called **outliers**. Then, the problem is also described as finding the maximum inlier set in $\mathbf{S}$, called the **optimal consensus** of discrete line fitting.



Deterministic method for discrete line fitting

In the line parameter space, each point $\mathbf{p}_i \in \mathbf{S}$ corresponds to the strip of height 1. Thus the set of all the strips of $\mathbf{S}$ divides the parameter space into regions (as preimages).





The problem is to find the region where the maximum number of strips pass.

Solution (Kenmochi et al. 2010)

By using the topological sweep algorithm (for arrangement of lines), we can visit all the regions with the time complexity $O(N^2)$ and the space complexity $O(N)$ where N is the number of points.

Probabilistic method for discrete line fitting (RANSAC)

If you know the probability information of data (you know how many outliers are contained approximately), then you can try **RANSAC (RANdom Sample Consensus)**.

A solution is obtained with a certain probability; if we increase this probability, we need more **iterations**, whose number can be fixed by the above probability information of data.

RANSAC (Fischler, Bolles, 1981)

Iteratively,

1. choose two points, and make a discrete line model from them;
2. count the number of inliers, and compare it to the current best one;
3. if it is better, then replace it as the current best.

Alternative robust line fitting: Hough transform

3D straight lines and their discretization

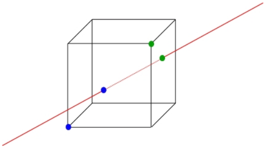
Definition (3D straight line)

A straight line in the Euclidean space $\mathbb{R}^3$ *is defined by*

$$\mathbf{L} = \{(\alpha_1 t + \beta_1, \alpha_2 t + \beta_2, \alpha_3 t + \beta_3) \in \mathbb{R}^3 : t \in \mathbb{R}\}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ *for* $i = 1, 2, 3$.

The *discretized line* $D(\mathbf{L})$ defined in $\mathbb{Z}^3$ by the *grid intersection* is the set of discrete points that are closest to the intersection in the plane of the *grid*.

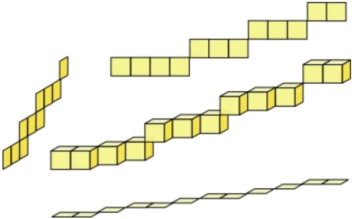


Discretized line and discrete curve

A discretized line is a 26-curve.

Theory (Kim, 1983)

A 26-curve is a discretized line if and only if two of its projections on the xy-, yz- and zx-planes are 8-connected 2D discrete lines.



The *3D discrete line recognition* is realized by the 2D discrete line recognition (three times).

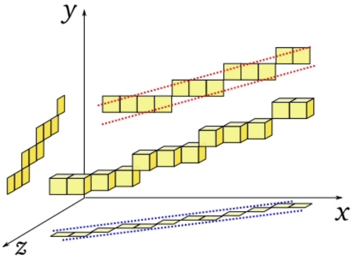
Arithmetic definition of 3D discrete lines

Definition (3D discrete line)

A set $G \subset \mathbb{Z}^3$ is an arithmetic line defined by seven integer parameters a, b, c, d_1, d_2, w_1, and w_2 if and only if

$$G = \{(x, y, z) \in \mathbb{Z}^3 : d_1 \leq cx - az < d_1 + w_1 \wedge d_2 \leq bx - ay < d_2 + w_2\}.$$

For simplification, we consider $0 \leq c \leq b \leq a$ and $\gcd(a, b, c) = 1$.



The parameters d_1 and d_2 are called the *lower bounds* and the parameters w_1 and w_2 define the *arithmetic thickness*

Thickness and connectedness of 3D discrete line

The thickness w_1 and w_2 allow to control the connectedness of the line.

Theory (Coeurjolly et al., 2001)

Let G be a discrete line defined by $a, b, c, d_1, d_2, w_1, w_2 \in \mathbb{Z}$ where $0 \leq c < b < a$, then:

1. *if $a + c \leq w_1$ and $a + b \leq w_2$, G is 6-connected;*
2. *if $a + c \leq w_1$ and $a \leq w_2 < a + b$, or if $a + b \leq w_2$ and $a \leq w_1 < a + c$, G is 18-connected;*
3. *if $a \leq w_1 < a + c$ and $a \leq w_2 < a + b$, G is 26-connected;*
4. *if $w_1 < a$ or $w_2 < a$, G is not connected.*

G is called a *3D naive line* if and only if $w_1 = w_2 = \max(|a|, |b|, |c|)$.

3D naive line

Theory (Coeurjolly et al., 2001)

A rational line discretized by the grid intersection is a 3D naive line and vice-versa.

According to Theory (Kim, 1983), we obtain the following corollary:

Corollary (Coeurjolly et al., 2001)

A 26-curve is a 3D naive line if and only if two of its projections on the xy-, yz- and zx-planes are 2D naives lines.

3D discrete line recognition

Problem (3D discrete line recognition)

Given a set of 3D discrete points $\mathbf{X}$*, do the points of* $\mathbf{X}$ *belong to a 3D discrete line?*

Yes *or* **No**

If yes, what are the parameters of this discrete line?

We apply the incremental algorithm for arithmetic line recognition (for a 2D naive line) (Debled-Rennesson and Reveillès, 1995) to each projection of $\mathbf{X}$ on the *xy*-, *yz*- and *zx*-planes.

If **Yes** for two of its projections, then **"Yes"**.

References

- I. Sivignon et I. Debled-Rennesson.
  "Droites et plans discrets," Chapitre 6 dans "Géométrie discrète et images numériques," Hermès, 2007.
- R. Klette and A. Rosenfeld.
  "2D Straightness", Chapter 9 in "Digital geometry: geometric methods for digital picture analysis," Morgan Kaufmann, 2004.