Master 2 “SIS”

Digital Geometry

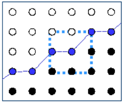
Topic 3:
Discrete surfaces and object boundaries:
from a grid point set to a polygon mesh

Yukiko Kenmochi

October 17, 2012

Aproaches to define discrete surfaces

- **Simple surface point approach** (Morghenthaler, Rosenfeld,1981; Couprie, Bertrand, 1998)



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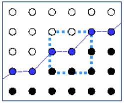
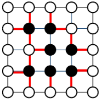
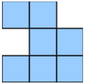


- **Adjacency graph approach** (Arzy, Frieder, Herman, 1981; Herman 1998)





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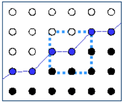

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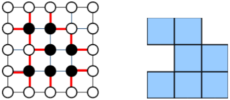
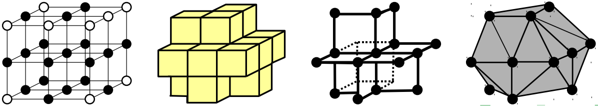


- **Cell complex approach** = **Mesh**
  - cubical complex (Kovalevsky, 1989; Khalimsky, 1990)
  - simplicial complex (Larensen, Cline, 1987; Lachaud, 2000)



Simple point and discrete curve

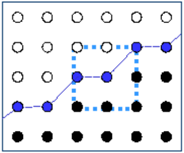
Definition (Simple point (Rosenfeld, 1973))

Given a $\mathbf{X} \subset \mathbb{Z}^n$, *let us consider* α*-connectedness for* $\mathbf{X}$ *and* β*-connectedness for* $\overline{\mathbf{X}}$. *Then, a point* $\mathbf{p} \in \mathbf{X}$ *is said to be* **simple** *if*

- $\mathbf{X}$ *and* $\mathbf{X} \setminus \{\mathbf{p}\}$ *have the same number of* α*-connected components;*
- $\overline{\mathbf{X}}$ *and* $\overline{\mathbf{X} \setminus \{\mathbf{p}\}}$ *have the same number of* β*-connected components.*

(α, β) must be a good pair: for example,

- $(4, 8), (8, 4)$ for $n = 2$,
- $(6, 18), (6, 26), (18, 6), (26, 6)$ for $n = 3$.



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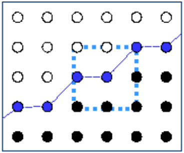

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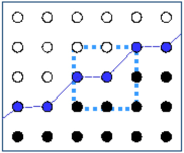

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Simple surface point = generalize Jordan curve theorem to 3D

Cell complex

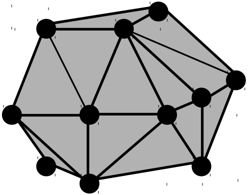
Definition (Cell complex)

A cell complex is a set C of cells such that

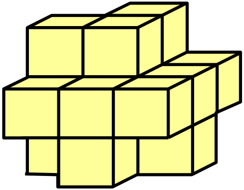
- *the empty cell is included in C,*
- *all the faces of every cell of C also belong to C,*
- *the intersection of two cells is one of their common faces.*

The r-cell is an r-dimensional convex polyhedron.

Simplicial complex



Cubical complex

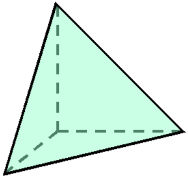


Face of complex

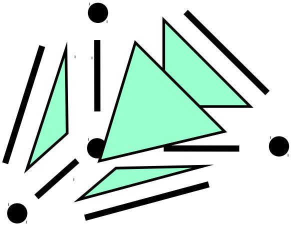
Definition (Face)

A face of an r-cell σ is an s-cell that is included in the boundary of σ with $s < r$.

3-cell

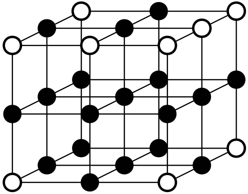


its faces

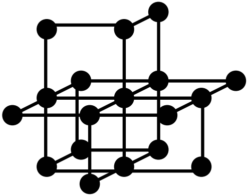


Digital image and complex representations

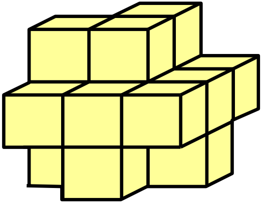
Grid point set
(digital image)



Adjacency graph



Kovalevsky topology
(cubical complex)



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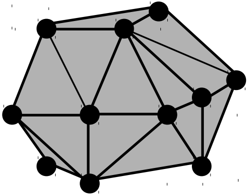
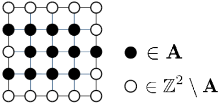
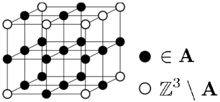


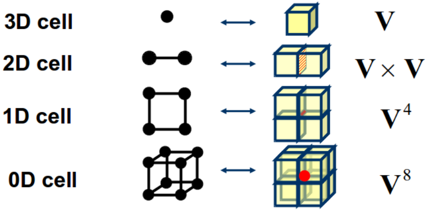
Kovalevsky topology

For a digital image,





we define cells as



Definition

Kovalevsky topology is defined by

$$C = (A, B)$$

such that A *is a set of cells and* $B \subset A \times A$ *is a set of their orders.*

Order of cells

cell order

If an r-cell σ is a face of s-cell τ, then

$$\sigma < \tau.$$

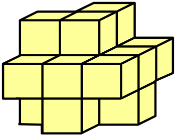
Note: $r < s$.

3D discrete object and boundary (Kovalevsky topology)

Definition (3D discrete object (3-complex))

Given an m-object $A \subset \mathbb{Z}^3$ for $m = 6, 18, 26$, we obtain the set K_3 of 3-cells whose centroid are the points of A. The 3D discrete object is then represented by the 3-complex given by

$$K = K_3 \cup \{\tau < \sigma : \sigma \in K_3\}.$$



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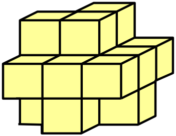

Definition (Boundary (2-cell set))

The boundary of a 3D discrete object in Kovalevky topology is the set of 2-cells *located between the interior border and the exterior border.*

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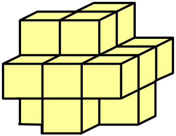

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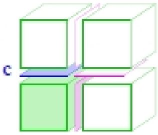
The boundary of a 3D discrete object in Kovalevky topology is the set of 2-cells located between the interior border and the exterior border.

This definition gives the same boundary as the **inter-voxel boundary** by using the adjacency graph.

Adjacency of 2-cells

Definition (2-cell adjacency)

Given a complex C, two distinct 2-cells of C are **adjacent** *if they have the common 1-face.*



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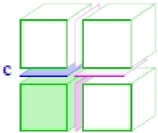
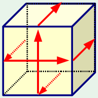


(Lachaud, Malgouyres, 2003)

Property (Voss, 1993)

In the boundary of a 6-object, each 2-cell is adjacent *to exactly* four neighboring 2-cells *such that two of them are its successors and the others are its predecessors.*

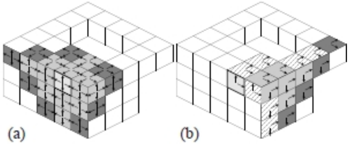


3D boundary following algorithm

Algorithm: 3D boundary following (Aztzy et al., 1981)

Input: 6-object, starting 2-cell s
Output: Set F of 2-cells that form the boundary

- Put s in a list F and in a queue Q, and also twice in a list L.
- **while** $Q \neq \emptyset$ **do**
  - Pull f from Q.
  - **for each** successor neighbor g of f **do**
    - **if** g is in L, pull g from L.
    - **otherwise** put g in F, in Q and in L.



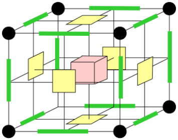
(Lachaud, Malgouyres, 2003)

The similar idea to the algorithm for connected component labeling is used.

Data structure for Kovalevsky topology

Data structure

If the size of input 3D image is $N \times N \times N$ (3D array size), the number of elements of its Kovalevsky topology is $2N \times 2N \times 2N = 8N^3$.



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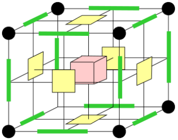

The dimension of each cell σ is determined by the 3D array index (i, j, k):

- if all of the integers i, j, k are even numbers, then σ is a 3-cell;
- if one of the integers i, j, k is odd, then σ is a 2-cell;
- if one of the integers i, j, k is even, then σ is a 1-cell;
- if all of the integers i, j, k are odd numbers, then σ is a 0-cell.

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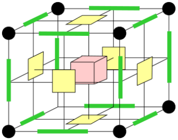

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The order between two cells is also defined depending on their 3D array indices.

Properties obtained in Kovalevsky topology

- Any m-object in the n-dimensional space is represented by an **n-complex** (the maximum dimension of its cells is n).

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- The boundary of a 6-object is a **combinatorial 2-manifold**.

  useful for geometric measurement

Combinatorial manifold

Definition (2-dimensional combinatorial manifold)

A pure 2-complex C is said to be **2-dimensional combinatorial manifold***, if*

- *every 1-cell of C is adjacent to exactly two 2-cells, and*
- *for every 0-cell v, the 2-cells each of which has v as its 0-face can be organized in a circular permutation $(f_0, f_1, \ldots, f_{k-1})$, $k > 1$, called the* umbrella *of v, such that for all i, f_i is adjacent to f_{i+1} (indices taken modulo k).*

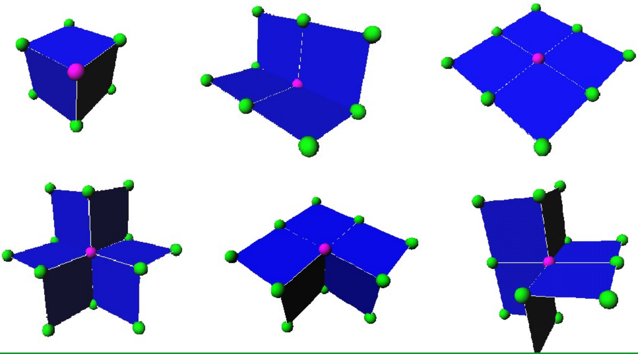
topologically equivalent to a disk

Local configurations of “cubical” 2-manifold

Property

Let us consider a cubical 2-complex C that is 2-manifold. Then, every 0-cell in C has one of the following local configurations.

(Françon, 1995)



Topological properties of discrete surfaces

We expect that **discrete surfaces** as discrete object boundaries have **correct topology**; for example, they are

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Question

Is there a discrete surface notion that allows to has the combinatorial manifold property for any connectedness?

Isosurface

If we accept the **inter-voxel boundary**, why not isosurface?

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Definition (Isosurface)

For a scalar function $f : \mathbb{R}^3 \leftarrow \mathbb{R}$*, we call* the isosurface of value s *the implicit surface defined by* $f(x, y, z) = s$*.*

Isosurface

If we accept the **inter-voxel boundary**, why not isosurface?

Definition (Isosurface)

For a scalar function $f : \mathbb{R}^3 \leftarrow \mathbb{R}$, *we call* the isosurface of value s *the implicit surface defined by* $f(x, y, z) = s$.

Marching cubes **algorithm** constructs the isosurface of value s with an *approximation of the function f* from the binary function

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \mathbf{A}, \\ 0 & \text{otherwise,} \end{cases}$$

or the gray-value function.

Marching cubes : construction of isosurface

Algorithm : *Marching cubes* (Lorensen, Cline, 1987)

Input: 3D image I
Output: Isosurface T

- **for each** unit cube in I, obtain triangles by referring to the *table of configurations* and put them in T.

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Example :

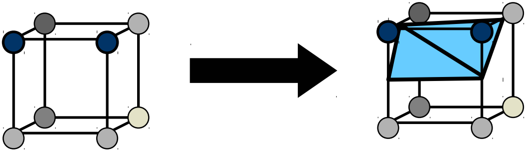
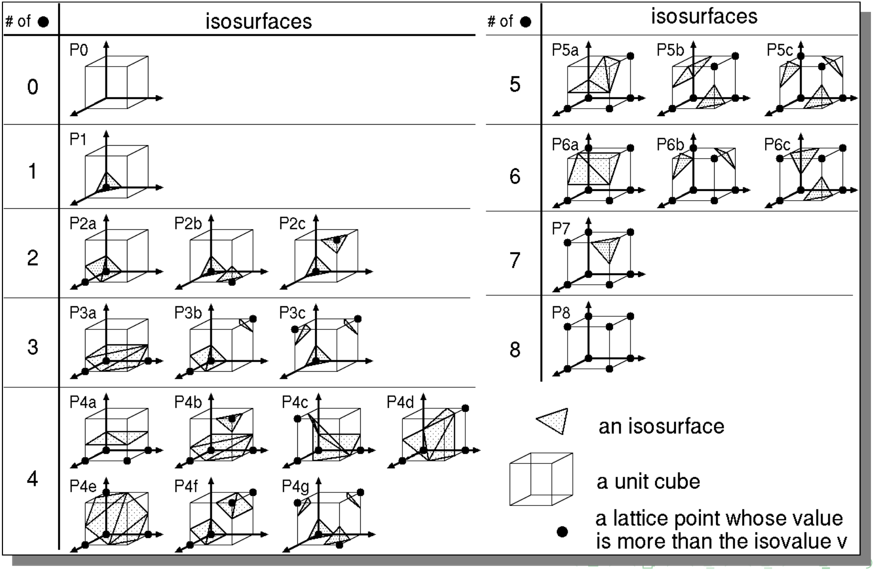
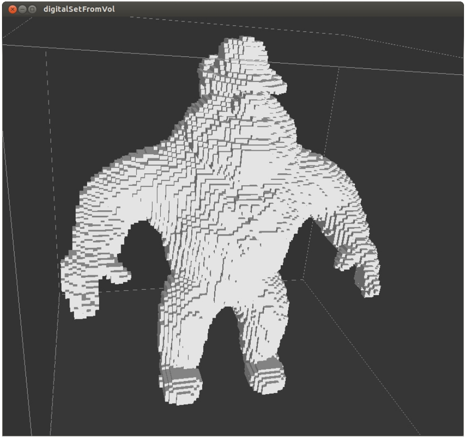


Table of configurations of *Marching cubes*

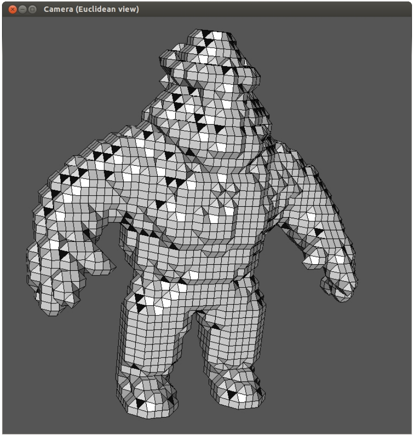


Results of *Marching cubes*

Voxel data: Al (Capone?)
(50×50 *size*)



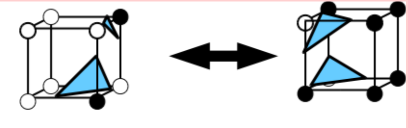
Isosurface



Problems of *Marching cubes*

We observe the following disadvantages:

- the original method (Lorensen, Cline, 1987) does not maintain topological guarantee (neither Jordan surface nor combinatorial manifold);



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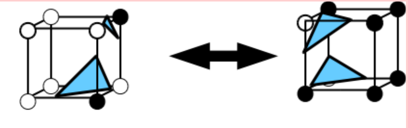
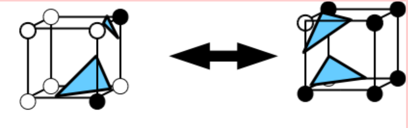


- the complexity is linear to the image size, $O(N^3)$, while that of 3D boundary following based on cubical complex is linear to the number of border points, $O(N^2)$.

Problems of *Marching cubes*

We observe the following disadvantages:

- the original method (Lorensen, Cline, 1987) does not maintain topological guarantee (neither Jordan surface nor combinatorial manifold);



- the complexity is linear to the image size, $O(N^3)$, while that of 3D boundary following based on cubical complex is linear to the number of border points, $O(N^2)$.

Both of them can be solved!

Continuous analog Jordan $\kappa\lambda$-boundary

The triangulated surface generated by the following table **guarantees the topology**. (Lachaud, Montanvert, 2000)

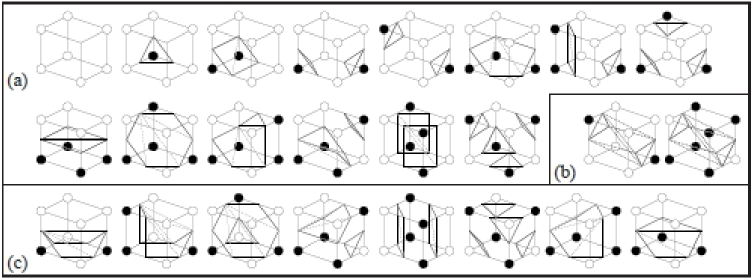


Figure 12.2. *Tables de configurations pour extraire une isosurface en fonction des connexités* (κ, λ) *choisies pour les 1-voxels et les 0-voxels. (a) configurations pour* $(\kappa, \lambda) \in \{(6, 18), (6, 26)\}$, *(b) Cas particulier pour* $(26, 6)$ *(son complémentaire est le cas particulier pour* $(6, 26)$. *(c) Si* $(\kappa, \lambda) \in \{(18, 6), (26, 6)\}$, *ces configurations sont triangulées ainsi.*

(Lachaud, Malgouyres, 2003)

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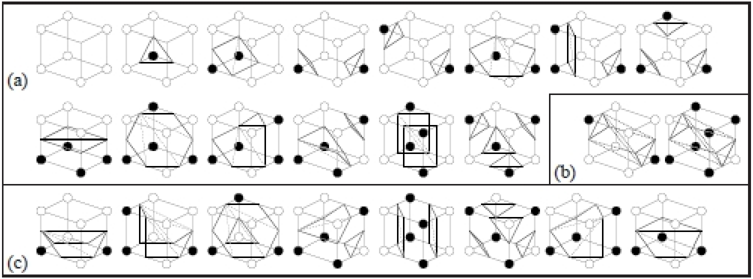


Figure 12.2. *Tables de configurations pour extraire une isosurface en fonction des connexités* (κ, λ) *choisies pour les 1-voxels et les 0-voxels. (a) configurations pour* $(\kappa, \lambda) \in \{(6,18),(6,26)\}$*, (b) Cas particulier pour* $(26,6)$ *(son complémentaire est le cas particulier pour* $(6,26)$*). (c) Si* $(\kappa, \lambda) \in \{(18,6),(26,6)\}$*, ces configurations sont triangulées ainsi.*

(Lachaud, Malgouyres, 2003)

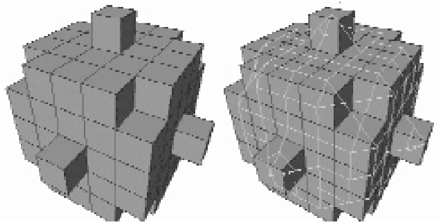
This still has a linear complexity $O(N^3)$ to the image size.

Duality between cubical complex boundary and $\kappa\lambda$-Jordan isosurface

Between **cubical complex boundary** B and $\kappa\lambda$-**Jordan isosurface** S, there are the following correspondences:

- 2-**cells of** B vs 0-**cells of** S,
- 1-**cells of** B vs 1-**cells of** S,
- 0-**cells of** B vs 2-**cells of** S,

which lead the **duality** between B and S.



(Lachaud, Valette, 2003)

Improvement of $\kappa\lambda$-Jordan isosurface

Complexity

Thanks to the duality, we can obtain the topological correct triangulated isosurface with a similar complexity to that of the 3D boundary following algorithm, $O(N^2)$, instead of $O(N^3)$.

Improvement of $\kappa\lambda$-Jordan isosurface

Complexity

Thanks to the duality, we can obtain the topological correct triangulated isosurface with a similar complexity to that of the 3D boundary following algorithm, $O(N^2)$, instead of $O(N^3)$.

Once you have a topologically correct mesh, then you can

- improve your mesh,
- compute geometrical and topological properties,
- visualize your object,
- deform your object, etc.

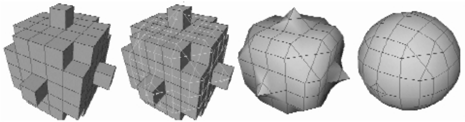


Figure 12.4. *Dualité surface discrète et isosurface : (a) bord discret d'une boule discrétisée, (b) graphe d'adjacence entre bels de cette surface discrète, (c) triangulation de ce graphe, (d) déplacement des bels/sommets avec l'équation de positionnement du MC.*

(Lachaud, Valette, 2003)

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  "Digital geometry: geometric methods for digital picture analysis", San Diego: Morgan Kaufmann, 2004.
- Jacques-Olivier Lachaud et Rémy Malgouyres.
  "Topologie, courves et surfaces discrètes", Chapitre 3 dans "Géométrie discrète et images numériques", Hermès Lavoisier, 2007.
- Jacques-Olivier Lachaud et Sébastien Valette.
  "Approximation par triangulation", Chapitre 12 dans "Géométrie discrète et images numériques", Hermès Lavoisier, 2007.