Master 2 “SIS”

Digital Geometry

Topic 5:
Geometric measurements of discrete shapes

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Shape geometric measurements

Example

Given a (2D) discrete object, we would like to estimate its

- **area**,
- **perimeter**,
- **tangent** *(field)*,
- **curvature** *(field)*,
- ...

Those geometric measuments are used for

- shape analysis,
- shape recognition,
- shape deformation,
- visualization,
- ...

Assumptions and basic notions

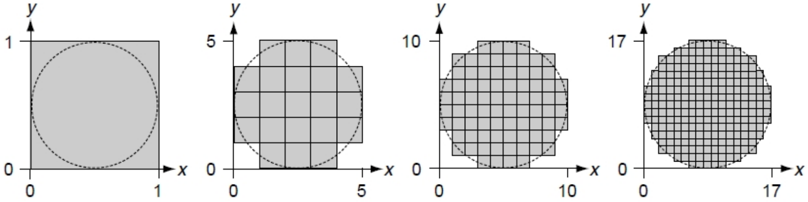
Mathematical setting

- Let S be a region (original object) in $\mathbb{R}^2$;
- γ be its boundary that is a Jordan curve;
- $h > 0$ be a grid resolution (discrete space: $\mathbb{Z}_h^2 = \{(\frac{i}{h}, \frac{j}{h}) : i, j \in \mathbb{Z}\}$);
- G_h be a Gauss discretization of S;
- γ_h be a closed m-curve ($m = 4$ or 8) that is
  - the $(8 - m)$-interior border of G_h, or
  - the inter-pixel boundary (it is considered as a closed 4-curve whose sequence elements are 0-cells.

Multigrid convergence

Multigrid convergence

Given an object $S \subset \mathbb{R}^2$, for each geometric estimator, we verify its multigrid (asymptotic) convergence; the estimated value E_h tends to the true value T when the image resolution h increases.



(Klette and Rosenfeld, 2003)

Multigrid convergence for global geometric features

For global geometric features, which are estimated from γ_h, such as perimeter, we use the following definition.

Multigrid convergence for global geometric feature

If $F(\gamma)$ is a global geometric feature of γ and E is an estimator of F, E is asymptotically convergent to F if and only if for any increasing resolution sequence h_i that tends to ∞, the sequence $E(\gamma_{h_i})$ converges to $F(\gamma)$.

For area estimation, we replace γ and γ_{h_i} by S and G_{h_i}.

Multigrid convergence for local geometric features

For local geometric features, which are calculated locally at each point of γ_h, such as tangent and curvature, we need to give a convergence definition point by point.

Point correspondence

A discrete point $\mathbf{x}_h$ is a h-discretization of a point $\mathbf{x}$ of γ if and only if $\|\mathbf{x} - \mathbf{x}_h\|_1 \leq \frac{1}{h}$ and $\mathbf{x}_h \in \gamma_h$.

Multigrid convergence for local geometric feature

If $F(\gamma, \mathbf{x})$ is a local geometric feature of γ at $\mathbf{x}$ and E is an estimator of F, E is asymptotically convergent to F if and only if for any increasing resolution sequence h_i that tends to ∞, for any point $\mathbf{x} \in \gamma$ having the h_i-discretization $\mathbf{x}_{h_i}$, the sequence $E(\gamma_{h_i}, \mathbf{x}_{h_i})$ converges to $F(\gamma, \mathbf{x})$.

Perimeter estimators

- local estimators
- estimators based on polygonalization by discrete lines
- tangent-based estimators

Local perimeter estimation (BLUE)

Statistic analysis is used to find weights that minimize the mean square error between the estimated and true length of a straight line segment.

Best linear unbiased estimator (Dorst, Smeulders, 1987)

Given a 8-curve γ_h, the perimeter estimator is

$$L_{BLUE}(\gamma_h) = \frac{1}{h}(0.948 n_i + 1.343 n_d)$$

where n_i is the number of isothetic steps and n_d the number of diagonal steps in γ_h.

Similar estimators have been proposed for chamfer distance using a 3×3 neighborhood (Borgefors, 1986):

$$L_{chamfer}(\gamma_h) = \frac{1}{h}(0.95509 n_i + 1.33693 n_d).$$

Local perimeter estimation (COC)

Corner-count estimator (Vossepoel, Smeulders, 1982)

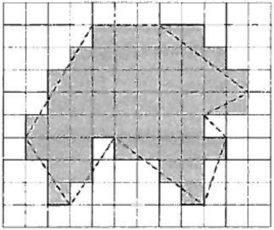
The perimeter estimator is

$$L_{COC}(\gamma_h) = \frac{1}{h}(0.980n_i + 1.406n_d - 0.091n_c)$$

where n_c is the number of odd-even transitions in the chain code of γ_h.

Perimeter estimation based on polygonalization

Polygonalization of a m-curve γ_h is a segmentation of γ_h into a set of discrete line segments.



(Feschet, Vialard, 2007)

Most probable original length estimation (Dorst, Smeulders, 1991)

For a 8-connected discrete line segment γ_h,

$$L_{MPO}(\gamma_h) = \frac{1}{h} n \sqrt{1 + \left(\frac{a}{b}\right)^2}$$

where n is the number of elements γ_h, $\frac{a}{b}$ is the best possible rational slope estimate.

Perimeter estimation based on polygonalization (cont.)

Polygonalization by discrete lines

We apply the arithmetic line recognition algorithm (Debled-Rennesson, Reveillès, 1995) to obtain an approximated polygon of γ_h, that is represented by a set of discrete line segments.

Note that the polygonalization is not uniquely defined: it depends on the method, the chosen starting point, and the direction in which the curve is traced.

MPO based perimeter estimation

For the perimeter estimation, we sum the MPO length estimates of the discrete line segments.

Tangent-based perimeter estimators

Curve length by integrating

Given a curve $\gamma(t) = (x(t), y(t))$ for $t \in [a, b]$, the tangent vector associated with $\gamma(t)$ is given by $\mathbf{n}(t) = (x'(t), y'(t))$ and then the curve length between $t = a$ and $t = b$ is

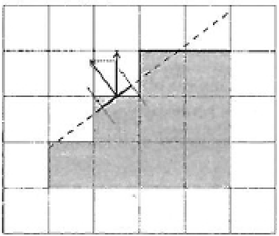
$$L(\gamma) = \int_a^b \|\mathbf{n}(t)\| \mathrm{d}t.$$

Tangent-based perimeter estimators (cont.)

Discrete curve length by integrating

Let us consider γ_h as a 1-complex. For each 1-cell e in γ_h, Let $\mathbf{n}(e)$ be the normal vector of e and $\hat{\mathbf{n}}(e)$ the estimated normal vector on e. Then, the length is estimated by

$$L_{TAN}(\gamma_h) = \sum_{e \in \gamma_h} \hat{\mathbf{n}}(e) \cdot \mathbf{n}(e).$$



(Feschet, Vialard, 2007)

Multigrid convergences of perimeter estimators

- **local estimators**:
  no multigrid convergence. (Tajine, Daurat, 2003)
- **estimators based on polygonalization by discrete lines**:
  multigrid convergence with a speed bounded by

  $$\frac{2\pi}{h}\left(\epsilon(h) + \frac{1}{\sqrt{2}}\right)$$

  where $\epsilon(h)$ corresponds to the distance between the discrete boundary and the approximated polygon (for example, $\frac{1}{h}$, depending on the algorithm). (Klette, Zunic, 2000)
  The proof is given for all polygonal, convex and r-compact sets.
- **tangent-based estimators**:
  if the tangent estimator converge asymptotically, then the perimeter estimator converge as well. (Coeurjolly, Klette, 2004)

Extension of length estimation to 3D

- **length measurement of a 3D discrete curve**:
  - local estimator based on curve-point configuration in a 26-neighborhood (Jonas, Kiryati, 1998);
  - estimator based on polygonal approximation of a 3D discrete curve, which is realized by applying the algorithm for recognizing 3D discrete line segments (Coeurjolly, et al. 2001);
- **surface area measurement of a 3D discrete surface**:
  - local estimator based on surface-point configuration in a 6-neighborhood (Mulkin, Verbeek, 1993);
  - triangulation methods for polyhedral approximation, which help to estimate the surface area.

Tangent estimators

- local estimators by using a fixed neighborhood of size $2k+1$,
- estimators by using adaptive neighborhoods.

Local tangent estimators

There are several tangent estimators by using a finite neighborhood of $2k+1$ points of a discrete curve around a point $\mathbf{x}_i$.

- **Median tangent** (Matas, Shao, Kittler, 1995):
  The tangent at $\mathbf{x}_i$ is estimated as the median direction of vectors $\overrightarrow{\mathbf{x}_i, \mathbf{x}_{i+j}}$ for $j = -k, \ldots, k$.
- **Average tangent** (Lenoir, Malgouyres, Revenu, 1996):
  The tangent at $\mathbf{x}_i$ is defined as the local average orientation and calculated by using a recursive Gaussian filter.
- **Best linear approximation tangent** (Anderson, Bezdek, 1984):
  The tangent at $\mathbf{x}_i$ is defined as the best approximation line of the neighborhood of $\mathbf{x}_i$ in the sense of minimizing the sum of squared distance of the $2k+1$ points.

Problem

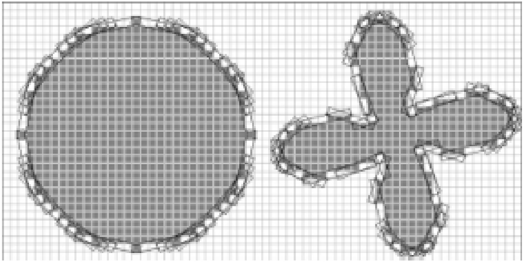
This approach does not allow to adapter the calculation to the local geometry of the curve.

Maximal segment

Adaptive-neighborhood based tangent estimator need the notion of maximal segment.

Maximal segment

The maximal segment is a sequence of points of the curve shaping a discrete line segment such that the discrete line segment cannot be extended by adding points of the curve to its endpoints.

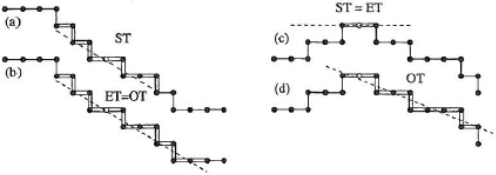


(Lachaud et al., 2007)

Tangent estimators based on adaptive neighborhoods

Discrete tangents

- **Symmetric tangent** at $\mathbf{x}_i$ is the longest discrete line segment with the form $\mathbf{x_{i-l}}, \ldots, \mathbf{x_{i+l}}$. (Lachaud, Vialard, 2003)
- **Oriented tangent** is the maximal segment with biggest indices that includes the symmetric tangent. Note that results depend on the orientation choice. (Feshet, Tougne, 1999)
- **Extended tangent** is obtained from the symmetric tangent; if it can be extended by either $\mathbf{x_{i-l-1}}$ or $\mathbf{x_{i+l+1}}$, it is equal to the symmetric tangent; otherwise, it is extended by as much as possible. (Braquelaire, Vialard, 1999)



(Feschet, Vialard, 2007)

Linear combination of adaptive-neighborhood tangents

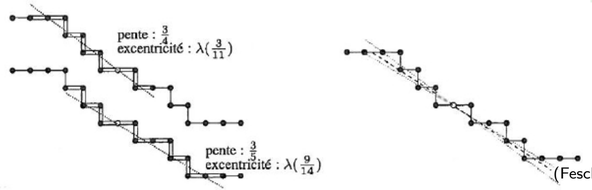
λ-MST (Lachaud, Vialard, de Vielleville, 2007)

The λ-MST estimator calculates the tangent direction θ of $\mathbf{x}_i$ as

$$\theta(\mathbf{x}_i) = \frac{\sum_{MS} e_{MS}(\mathbf{x}_i)\theta_{MS}}{\sum_{MS} e_{MS}(\mathbf{x}_i)}$$

where $e_{MS}(\mathbf{x}_i)$ is the eccentricity for $\mathbf{x}_i$ with respect to each maximal segment MS, defined by

$$e_{MS}(\mathbf{x}_i) = \begin{cases} \lambda\left(\frac{\|\mathbf{x}_i - \mathbf{x}_k\|_1}{\|\mathbf{x}_l - \mathbf{x}_k\|_1}\right) & \text{if } \mathbf{x}_i \in MS(= \mathbf{x}_k, \ldots, \mathbf{x}_l), \\ 0 & \text{otherwise.} \end{cases}$$



(Feschet, Vialard, 2007)

λ-MST algorithm

The λ-MST is based on the incremental algorithm for maximal segments of a discrete curve, whose time complexity is linear.

Algorithm: incremental algorithm of maximal segments

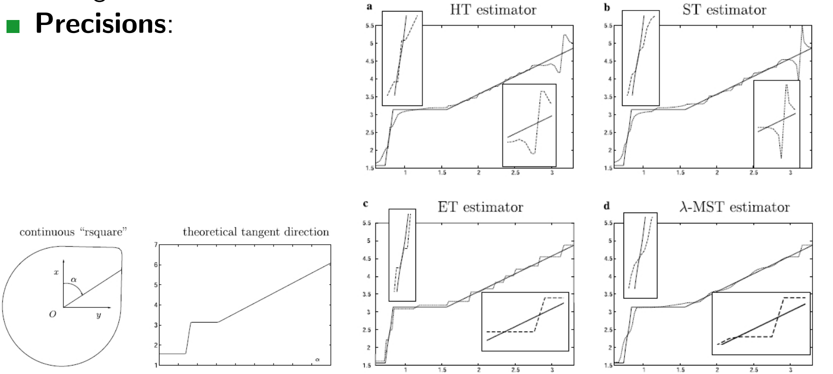
Input: discrete curve γ_h, maximal segment $(\mathbf{x}_k, \ldots, \mathbf{x}_l)$
Output: next maximal segment

- $k = k + 1;$
- $l = l + 1;$
- **while** $\neg S(k, l)$ **do** $k = k + 1$
- **while** $S(k, l)$ **do** $l = l + 1$
- $l = l - 1;$

Note that $S(i, j)$ denotes that the sequence $(\mathbf{x}_i, \ldots, \mathbf{x}_j)$ is a discrete line segment.

Comparisons of tangent estimators

- **Multigrid convergence**: Oriented tangent, λ-MST
  with a speed of average convergence bounded by $O(h^{-\frac{1}{3}})$
  The proof is given for a convex and three-times differentiable border having continuous curvature.
- **Precisions**:



(Lachaud et al., 2007)

The best choice may be the λ-MST.

Curvature estimators

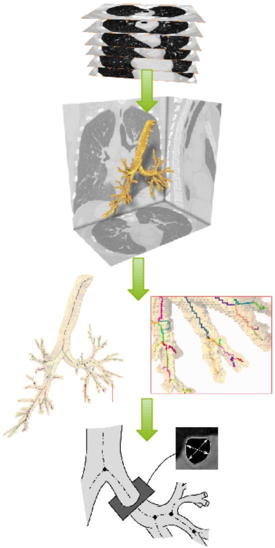
There are mainly three approaches: curvature is estimated from

- the change in the slope angle of the tangent line;
- derivatives along the curve;
- the radius of the osculating circle.

Further info can be found in the references.

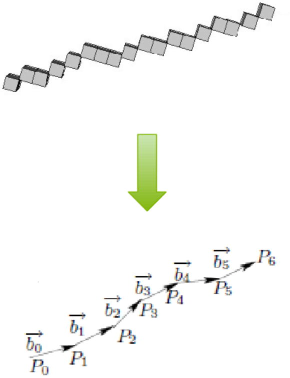
Application example

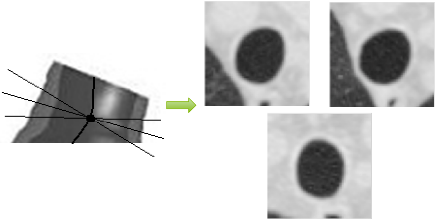
For supporting the non-invasive diagnosis of bronchial tree pathologies, automatic quantitative description of an airway tree extracted from volumetric CT data set is useful. (M. Postolski, 2011)



Application example: purpose

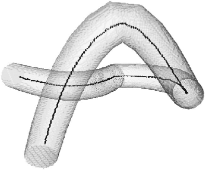
Tangent estimation from a 3D discrete curve.



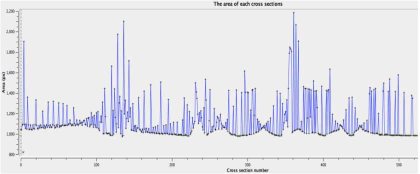


Application example: experiment 1

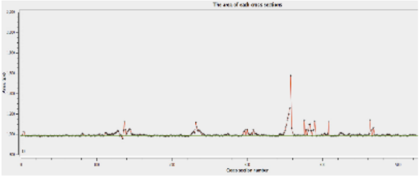
Experiments to a 3D tube.



Naive method



λ-MST



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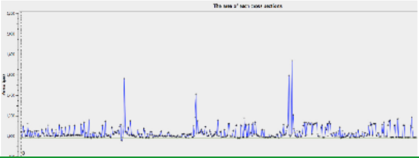
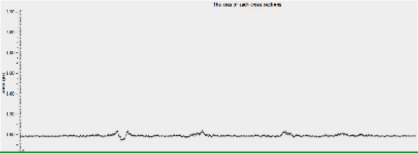
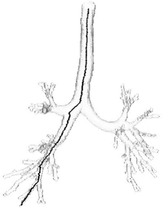


Spline

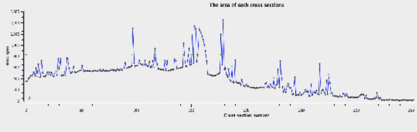


Application example: experiment 2

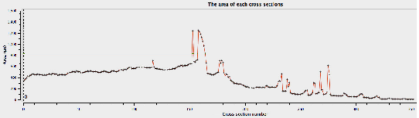
Experiments to a bronchial tree



Naive method



λ-MST



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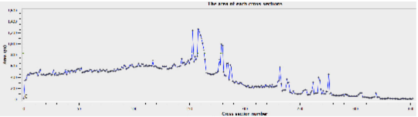
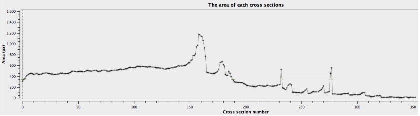


Spline



References

- F. Feschet et A. Vialard.
  "Attributs de bord," Chapitre 13 dans "Géométrie discrète et images numériques," Hermès, 2007.
- R. Klette and A. Rosenfeld.
  "2D arc length; curvature and corners", Chapter 10 in "Digital geometry: geometric methods for digital picture analysis," Morgan Kaufmann, 2004.
- J.-O. Lachaud, A. Vialard, and F. de Vieilleville.
  "Fast, accurate and convergent tangent estimation on digital contours," Image and Vision Computing, Vol. 25, pp. 1572–1587, 2007.