Master 2 “SIS”

# Digital Geometry

## Topic 4:
## Discrete lines and planes

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# Straight line

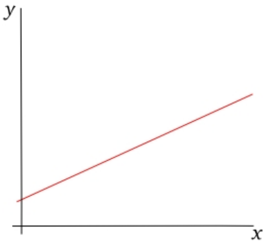
## Definition (Straight line)

*A line in the Euclidean space* $\mathbb{R}^2$ *is defined by*

$$\mathbf{L} = \{(x, y) \in \mathbb{R}^2 : \alpha x + \beta y + \gamma = 0\}$$

*where* $\alpha, \beta, \gamma \in \mathbb{R}$*.*

In general, we have a normalization such that $|\alpha| + |\beta| = 1$, $\alpha^2 + \beta^2 = 1$.



# Discretization of straight line

## Definition (Discrete line)

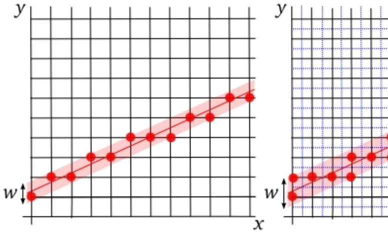
*The discrete line of* $\mathbf{L}$ *in* $\mathbb{Z}^2$ *is defined by*

$$D(\mathbf{L}) = \{(p, q) \in \mathbb{Z}^2 : 0 \leq \alpha p + \beta q + \gamma' \leq \omega\}$$

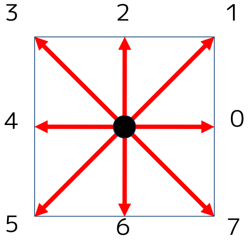
*where* $\omega$ *is called the* thickness.

The values of $\gamma'$ and $\omega$ depend on the model of discretization.

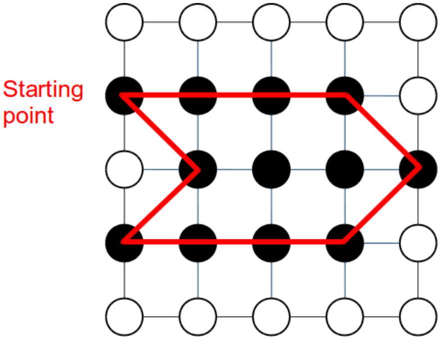
- **Grid-intersection**: grid points closest to the intersections with the grid lines
  $\gamma' = \gamma + \frac{\max(|\alpha|,|\beta|)}{2}$, $\omega = \max(|\alpha|, |\beta|)$.
- **Super-cover (outer Jordan)**: 2-cells intersecting with the line
  $\gamma' = \gamma + \frac{|\alpha|+|\beta|+1}{2}$, $\omega = |\alpha| + |\beta| + 1$.
- **Gauss** (half-plane): 2-cells with center points in the half-plane
  $\gamma' = \gamma$, $\omega$ decides the $m$-connectedness



# Freeman code




**Freeman code**



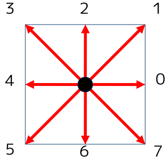

**Chain code:** 7500013444

# Properties of discrete lines

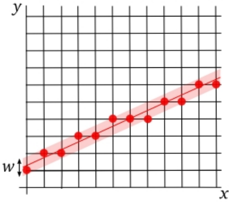
## Criteria of Freeman (1974)

For discrete lines (by *grid-intersection* discretization), the **Freeman code** verify the following three properties:

1. the code contains **at most two different values**;
2. those two values differ **at most by one unit** (modulo 8) ;
3. one of the two values **appears isolatedly** and its appearances are uniformly spaced in the code.




**Freeman code**




**Discrete line :** 10101001010...

# Properties of discrete lines (cont.)

## Definition (Chord property (Rosenfeld, 1974))

*A set of discrete points* $\mathbf{X}$ *satisfies the* chord property *if for every pair of points* $\mathbf{p}$ *and* $\mathbf{q}$ *of* $\mathbf{X}$ *and for every point* $\mathbf{r} = (r_x, r_y)$ *of the real segment between* $\mathbf{p}$ *and* $\mathbf{q}$*, there exists a point* $\mathbf{s} = (s_x, s_y)$ *of* $\mathbf{X}$ *such that* $\max(|s_x - r_x|, |s_y - r_y|) < 1$.

- It proves that a discrete curve is a discrete line segment if and only if it owns the chord property.
- It allows to show the two first criteria of Freeman and to deduce a number of properties that specify the third criterion.
- There are a number of algorithms for recognizing a discrete straight line based on this property.

# Bresenham line-drawing algorithm

## Algorithm: drawing a discrete line (Bresenham, 1962)

**Input:** Two discrete points $(x_1, y_1)$, $(x_2, y_2)$ (s.t. $x_2 - x_1 \geq y_2 - y_1 > 0$)
**Output:** Line segment between the two points

- $d_x = x_2 - x_1$, $d_y = y_2 - y_1$;
- $y = y_1$; initialization
- $e = d_x$; value of initial error
- **for** $x$ **from** $x_1$ **to** $x_2$ **do**
  - put the pixel $(x, y)$;
  - $e = e + 2d_y$;
  - **if** $e \geq 2d_x$ **then**
    - $y = y + 1$;
    - $e = e - 2d_x$;

Can we consider **rounding** instead of truncation?

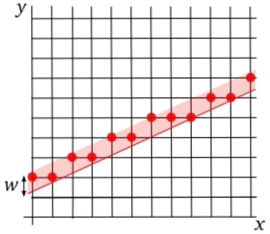
# Arithmetic definition of discrete lines

## Definition (Arithmetic discrete line)

*A discrete line of parameters* $(a, b, c)$ *and of arithmetic thickness* $w$ *where* $a, b, c \in \mathbb{Z}$ *and* $gcd(a, b) = 1$ *is defined as*

$$D(a, b, c, w) = \{(p, q) \in \mathbb{Z}^2 : 0 \leq ap + bq + c < w\}.$$

The thickness parameter $w$ allows to control the connectedness of the line.

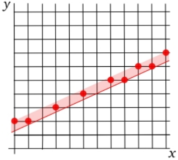


# Thickness and connectedness of discrete lines

## Theory

*Let $D(a, b, c, w)$ be a discrete line, then:*

1. *if $w < \max(|a|, |b|)$, it is not connected;*
2. *if $w = \max(|a|, |b|)$, it is a 8-curve ;* naive line
3. *if $\max(|a|, |b|) < w < |a| + |b|$, it is a $*$-curve (its two successive points are 4-neighboring or strictly 8-neighboring);*
4. *if $w = |a| + |b|$, it is a 4-curve;* standard line
5. *if $w > |a| + |b|$, it is said* thick.

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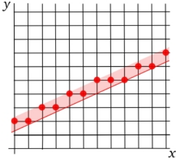


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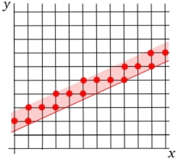


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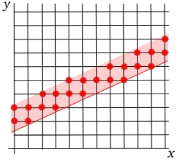


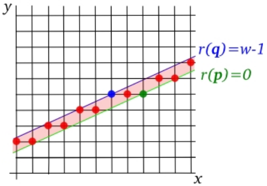

*5*

# Remainder and leaning point of discrete lines

## Definition (Remainder)

*The* remainder *associated to a point* $\mathbf{p} = (p_x, p_y)$ *of* $D(a, b, c, w)$ *is an integer value defined by*

$$R(\mathbf{p}) = ap_x + bp_y + c.$$




- When the remainder is 0, $\mathbf{p}$ is called a **lower leaning point**.
- When the remainder is $w - 1$, $\mathbf{p}$ is called a **upper leaning point**.

We can generalize the Bresenham algorithm by using the *remainder* instead of the *error e*.

# Arithmetic line drawing algorithm

## Algorithm: drawing an arithmetic (naive) line

**Input:** Two discrete points $(x_1, y_1)$, $(x_2, y_2)$ and $c$
**Output:** Line segment between the two points

- $b = x_2 - x_1$, $a = y_2 - y_1$;
- $y = y_1$;
- $r = ax_1 + by_1 + c$;
- **for** $x$ **from** $x_1$ **to** $x_2$
  - put the pixel $(x, y)$;
  - $r = r + a$;
  - **if** $r \geq b$ **then**
    - $y = y + 1$;
    - $r = r - b$;

We consider here that $x_2 - x_1 \geq y_2 - y_1 > 0$.

The value of $c$ is initially chosen such that $0 \leq ax_1 + by_1 + c < b$.

# Discrete line recognition

## Problem (Discrete line recognition)

*Given a set of discrete points* $\mathbf{X}$*, do the points of* $\mathbf{X}$ *belong to a discrete line?*

**Yes** *or* **No**

*If yes, what are the parameters of this discrete line?*

There are many recognition algorithms with linear complexity.

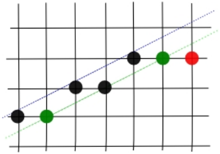
1. **approach of linear programming**:
   verify the existence of feasible (real) solutions.
2. **approach based on preimage** (Lindenbaum, Bruckstein, 1993):
   use the properties of discrete lines in the dual space, called preimages.
3. **arithmetic approach** (Debled-Rennesson, Reveillès, 1995):
   verify the existence of integer solutions by using arithmetic properties.
4. ...

# Incremental algorithm for arithmetic line recognition: initial situation

Let

- $\mathbf{S}$ be a segment of naive line $D(a, b, c)$ with $0 \leq a < b$,
- $\mathbf{q} = (x_\mathbf{q}, y_\mathbf{q})$ be the point of the greatest abscissa of $\mathbf{S}$,
- $\mathbf{l}$ and $\mathbf{l'}$ be the lower leaning points of minimum and maximum abscissas of $\mathbf{S}$,
- $\mathbf{u}$ and $\mathbf{u'}$ be the upper leaning points of minimum and maximum abscissas of $\mathbf{S}$.

By adding a point $\mathbf{p} = (x_\mathbf{p}, y_\mathbf{p})$ connected to $\mathbf{S}$ such that $x_\mathbf{p} = x_\mathbf{q} + 1$, we verify if $\mathbf{S}' = \mathbf{S} \cup \{\mathbf{p}\}$ is still a naive line segment.

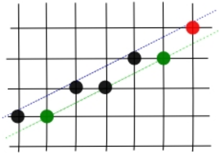


# Incremental algorithm for arithmetic line recognition

## Theory (Debled-Rennesson and Reveillès, 1995)

*We have*

1. *if* $0 < r(\mathbf{p}) < b$, $\mathbf{S}'$ *is a naive line segment* $D(a, b, c)$;
2. *if* $r(\mathbf{p}) < -1$ *or* $b < r(\mathbf{p})$, *then* $\mathbf{S}'$ *is not a naive line segment;*
3. *if* $r(\mathbf{p}) = -1$, *then* $\mathbf{S}'$ *is a naive line segment* $D(y_{\mathbf{p}} - y_{\mathbf{u}}, x_{\mathbf{p}} - x_{\mathbf{u}}, -ax_{\mathbf{p}} + by_{\mathbf{p}})$;
4. *if* $r(\mathbf{p}) = b$, *then* $\mathbf{S}'$ *is a naive line segment* $D(y_{\mathbf{p}} - y_{\mathbf{l}}, x_{\mathbf{p}} - x_{\mathbf{l}}, -ax_{\mathbf{p}} + by_{\mathbf{p}} + b - 1)$.



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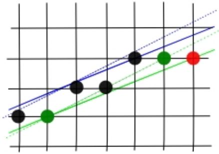

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# Farey sequence

## Definition (Farey sequence (Hardy and Write, 1979))

*The Farey sequence of order $n$, $F_n$, is the sequence of irreducible fractions between 0 and 1, whose denominators are less than or equal to $n$, in ascending order.*

If $0 <= h <= k <= n$ and $\gcd(h, k) = 1$, then $\frac{h}{k}$ is in $F_n$.

## Example ($F_5$)

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

# Structure of the Farey sequence: Stern-Brocot tree

## Property (Neighborhood)

*If $\frac{a}{b}$ and $\frac{c}{d}$ are neighboring in a Farey sequence, with $\frac{a}{b} < \frac{c}{d}$, then their difference is equal to $\frac{1}{bd}$.*

## Property (Median)

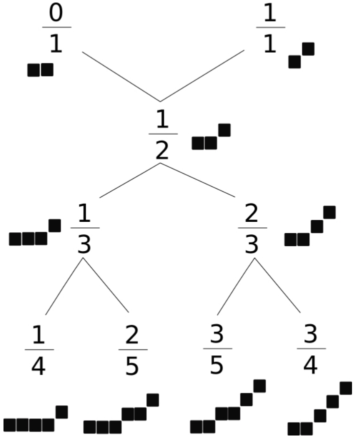
*If $\frac{a}{b}$, $\frac{p}{q}$ and $\frac{c}{d}$ are neighboring in a Farey sequence such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$, then $\frac{p}{q}$ is the median of $\frac{a}{b}$ and $\frac{c}{d}$ such as*

$$\frac{p}{q} = \frac{a+c}{b+d}.$$

These properties allow to construct the *Stern-Brocot tree*.

# Stern-Brocot tree and discrete lines

Each vertex $\frac{h}{k}$ of the tree corresponds to a pattern (motif) associated to the discrete line of slope $\frac{h}{k}$.




Updating parameters of the incremental discrete line recognition algorithm indicates moving from the tree root to a leaf.

# Applications of discrete line recognition

The discrete line recognition allow us to:

- study the *parallelism*, *colinearity*, *orthogonality*, *convexity* in the discrete space;
- estimate geometric properties of discrete object borders, such as the *length* of a curve, *tangent* and *curvature* at a point in a curve, etc.;
- make a segmentation of a discrete curve into line segments (*polygonalisation*).

**If there is noise in discrete object border, we need to modify the problem.**

# 3D straight lines and their discretization

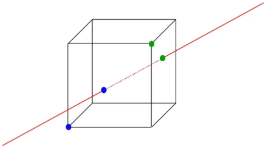
## Definition (3D straight line)

*A straight line in the Euclidean space* $\mathbb{R}^3$ *is defined by*

$$\mathbf{L} = \{(\alpha_1 t + \beta_1, \alpha_2 t + \beta_2, \alpha_3 t + \beta_3) \in \mathbb{R}^3 : t \in \mathbb{R}\}$$

*where* $\alpha_i, \beta_i \in \mathbb{R}$ *for* $i = 1, 2, 3$.

The *discretized line* $D(\mathbf{L})$ defined in $\mathbb{Z}^3$ by the *grid intersection* is the set of discrete points that are closest to the intersection in the plane of the *grid*.



# Discretized line and discrete curve

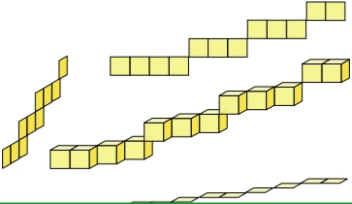
A discretized line is a 26-curve.

## Definition ($m$-curve)

*An m-path $\pi$ is an m-curve if for every element $\mathbf{p}_i$ of $\pi$, $i = 1, \ldots, n$, $\mathbf{p}_i$ has exactly two m-adjacent points in $\pi$, except for $\mathbf{p}_0$ and $\mathbf{p}_n$ that has only one.*

## Theory (Kim, 1983)

*A 26-curve is a discretized line if and only if two of its projections on the xy-, yz- and zx-planes are 8-connected 2D discrete lines.*



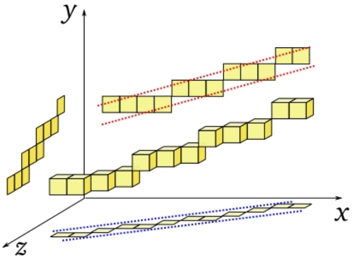
# Arithmetic definition of 3D discrete lines

## Definition (3D discrete line)

*A set $G \subset \mathbb{Z}^3$ is an arithmetic line defined by seven integer parameters $a, b, c, d_1, d_2, w_1$, and $w_2$ if and only if*

$$G = \{(x, y, z) \in \mathbb{Z}^3 : d_1 \leq cx - az < d_1 + w_1 \wedge d_2 \leq bx - ay < d_2 + w_2\}.$$

For simplification, we consider $0 \leq c \leq b \leq a$ and $\gcd(a, b, c) = 1$.




The parameters $d_1$ and $d_2$ are called the *lower bounds* and the parameters $w_1$ and $w_2$ define the *arithmetic thickness*

# Thickness and connectedness of 3D discrete line

The thickness $w_1$ and $w_2$ allow to control the connectedness of the line.

## Theory (Coeurjolly et al., 2001)

*Let $G$ be a discrete line defined by $a, b, c, d_1, d_2, w_1, w_2 \in \mathbb{Z}$ where $0 \leq c < b < a$, then:*

1. *if $a + c \leq w_1$ and $a + b \leq w_2$, $G$ is 6-connected;*
2. *if $a + c \leq w_1$ and $a \leq w_2 < a + b$, or if $a + b \leq w_2$ and $a \leq w_1 < a + c$, $G$ is 18-connected;*
3. *if $a \leq w_1 < a + c$ and $a \leq w_2 < a + b$, $G$ is 26-connected;*
4. *if $w_1 < a$ or $w_2 < a$, $G$ is not connected.*

$G$ is called a *3D naive line* if and only if $w_1 = w_2 = \max(|a|, |b|, |c|)$.

# 3D naive line

**Theory (Coeurjolly et al., 2001)**

*A rational line discretized by the grid intersection is a 3D naive line and vice-versa.*

According to Theory (Kim, 1983), we obtain the following corollary:

**Corollary (Coeurjolly et al., 2001)**

*A 26-curve is a 3D naive line if and only if two of its projections on the xy-, yz- and zx-planes are 2D naives lines.*

# 3D discrete line recognition

**Problem (3D discrete line recognition)**

*Given a set of 3D discrete points* $\mathbf{X}$*, do the points of* $\mathbf{X}$ *belong to a 3D discrete line?*

**Yes** *or* **No**

*If yes, what are the parameters of this discrete line?*

We apply the incremental algorithm for arithmetic line recognition (for a 2D naive line) (Debled-Rennesson and Reveillès, 1995) to each projection of $\mathbf{X}$ on the *xy*-, *yz*- and *zx*-planes.

If **Yes** for two of its projections, then **"Yes"**.

# Euclidean plane

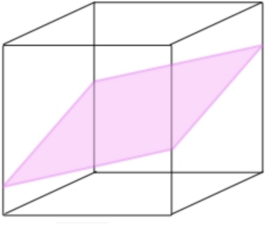
## Definition (Plane)

*A plane in the Euclidean space* $\mathbb{R}^3$ *is defined by*

$$\mathbf{P} = \{(x, y, z) \in \mathbb{R}^3 : \alpha x + \beta y + \gamma z + \delta = 0\}$$

*where* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

In general, we have a normalisation such that $|\alpha| + |\beta| + |\gamma| = 1$, $\alpha^2 + \beta^2 + \gamma^2 = 1$.



# Discretization of planes

**Definition (Discretized plane)**

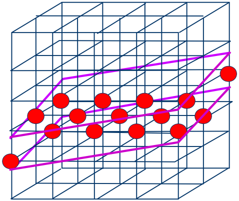
*The discretized plane of* $\mathbf{P}$ *in* $\mathbb{Z}^2$ *is defined by*

$$D(\mathbf{P}) = \{(p, q, r) \in \mathbb{Z}^2 : 0 \leq \alpha p + \beta q + \gamma r + \delta' \leq \omega\}$$

*where* $\omega$ *is called the* thickness.

The values of $\delta'$ and $\omega$ depend on the discretization model.

- **Grid intersection**: grid points closest to the intersections with the grid planes
  $\delta' = \delta + \frac{\max(|\alpha|,|\beta|,|\gamma|)}{2}$,
  $\omega = \max(|\alpha|, |\beta|, |\gamma|)$.
- **Super-cover (outer Jordan)**: 3-cells intersecting with the line
  $\delta' = \delta + \frac{|\alpha|+|\beta|+|\gamma|+1}{2}$,
  $\omega = |\alpha| + |\beta| + |\gamma| + 1$.
- **Gauss** (half-space): 3-cells with center



# Chordal triangle property

**Definition (Kim, 1984)**

*A set of 3D discrete points* $\mathbf{X}$ *satisfies the* chordal triangle property *if and only if for any triplet of points* $\mathbf{p}_1$, $\mathbf{p}_2$ *and* $\mathbf{p}_3$ *of* $\mathbf{X}$, *every point on the triangle* $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \in \mathbb{R}^3$ *is at* $L_\infty$*-distance* $< 1$ *from some point of* $\mathbf{X}$.

- This is an extension of Rosenfeld's chord property for 2D discrete lines.
- This is neither a necessary condition nor a sufficient condition for a piece of discrete surface to be a piece of discrete plane.

# Characterization based on the convex hull

This is the corrected version whose original one was proposed by Kim.

**Theory (Debled-Rennesson, 1995)**

*A set of discrete points* $\mathbf{X}$ *is a piece of discrete plane if and only if*

- *there exists a face* $F$ *of the convex hull* $conv(\mathbf{X})$ *of* $\mathbf{X}$ *such that the distance between* $\mathbf{X}$ *and the supporting plane of* $F$ *is less than* 1*, or*
- *there exist two edges* $A_1$ *and* $A_2$ *of* $conv(\mathbf{X})$ *such that the distance between* $\mathbf{X}$ *and the plane generated by* $A_1$ *and* $A_2$ *is less than* 1*.*

There exists an arithmetic algorithm for discrete plane recognition based on this theorem (Debled-Rennesson, 1995); however, its complexity is not analyzable.

The algorithm based on the original characterization of Kim has a complexity $O(n^4)$ where $n$ is the size of $\mathbf{X}$.

# Notion of separating plane

**Theory (Stojmentović and Tosić, 1991)**

*A set of discrete points* $\mathbf{X}$ *is a piece of discrete plane if and only if there exists an* Euclidean plane $\mathbf{P}$ that separates $\mathbf{X}$ and the set $\mathbf{X}'$ *that is obtained by translating* $\mathbf{X}$ *by* 1 *along one of the* $x$*-,* $y$*- and* $z$*-axes (this axis is called the principal axis of the plane).*

The algorithm based on this theorem has a complexity $O(n)$ by using techniques of linear programming. However, it is not incremental.

The incremental algorithm by using linear programming techniques was proposed and has a complexity $O(n)$ (Buzer, 2003).

# Evenness property

The property for the discrete lines (Hung, 1985) is extended to hyperplanes of arbitrary dimensions.

For simplification, we consider the planes with $0 \leq \beta \leq \gamma$ and $\gamma \neq 0$.

## Definition (Veelaert, 1993)

*A set of discrete points* $\mathbf{X}$ *is said even if and only if*

- *the projection of* $\mathbf{X}$ *on the plane* $z = 0$ *is bijective,*
- *for every quadruplet of points* $\mathbf{p_i} = (x_i, y_i)$, $i = 1, 2, \ldots, 4$, *of* $\mathbf{X}$ *such that* $x_1 - x_2 = x_3 - x_4$ *and* $y_1 - y_2 = y_3 - y_4$, *then* $|(z_1 - z_2) - (z_3 - z_4)| \leq 1$.

- This is necessary and sufficient to characterize infinite discrete planes and pieces of rectangular planes.
- This criterion can be evaluated in $O(n^2)$, with $n$ the size of $\mathbf{X}$.

# Algorithms for discrete plane recognition

1. **approach based on the linear programming**:
   $O(n)$ (Stojmenović and Tosić, 1991)
   $O(n)$ for an incremental algorithm (Buzer, 2003)
2. **approach based on the convex hull**:
   $O(n^7)$ with a linear behavior in practice (Gérard et al., 2005)
3. **approach based on the evenness**:
   $O(n^2)$ (Veelaert, 1994)
4. **arithmetic approach**:
   ? (Debled-Renesson and Reveillès, 1994)
5. **approach based on the preimage**:
   $O(n^3 \log n)$ (Vittone and Chassary, 2000)
6. ...

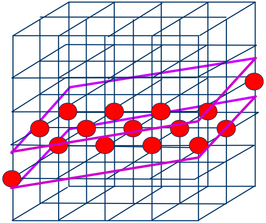
# Arithmetic definition of discrete planes

**Definition (Arithmetic plane (Reveillès, 1991))**

*A discrete plane of normal vector $(a, b, c)$ with translation parameter $d$ and arithmetic thickness $w$ where $a, b, c, d, w \in \mathbb{Z}$ and $gcd(a, b, c) = 1$ is defined such that*

$$\Pi(a, b, c, d, w) = \{(p, q, r) \in \mathbb{Z}^3 : 0 \leq ap + bq + cr + d < w\}.$$

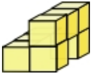
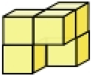
The thickness parameter $w$ allows to control the connectedness of the plane.



# Thickness and topology of discrete plane

## Definition ($m$-tunnel (Andres et al., 1997))

*A discrete plane* $\Pi(a, b, c, d, w)$ *has an* $m$-tunnel *if there exist two m-neighbors* $\mathbf{p}_A = (x_A, y_A, z_A)$ *and* $\mathbf{p}_B = (x_B, y_B, z_B)$ *such that* $ax_a + by_A + cz_A + d < 0$ *and* $ax_B + by_B + cz_B + d \geq w$.

6-*tunnel*  18-*tunnel*  26-*tunnel* 

## Theory (Andres et al., 1997)

*Let* $\Pi(a, b, c, d, w)$ *be a discrete plane such that* $0 \leq a \leq b \leq c$ *and* $c \neq 0$, *then:*

1. *if* $w < c$, $\Pi$ *has* 6-*tunnels ;*
2. *if* $c \leq w < b + c$, $\Pi$ *has* 18-*tunnels ;*
3. *if* $b + c \leq w < a + b + c$, $\Pi$ *has* 26-*tunnels ;*
4. *if* $a + b + c \geq w$, $\Pi$ *has no tunnel.*

# Thickness and connectivity of discrete planes

## Corollary (Andres et al., 1997)

*Let* $\Pi(a, b, c, d, w)$ *be a discrete plane such that* $0 \leq a \leq b \leq c$ *and* $c \neq 0$*, then:*

1. *if* $w = c$*,* $\Pi$ *is* 18*-connected;*
2. *if* $c < w < b + c$*,* $\Pi$ *is* 18*- or* 6*-connected;*
3. *if* $b + c \leq w$*,* $\Pi$ *is* 6*-connected.*

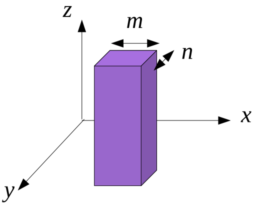
- We call *naive planes* the planes of thickness $w = \max(|a|, |b|, |c|)$, and *standard planes* the planes of thickness $w = |a| + |b| + |c|$.
- The naive planes are thus the finest 18-connected planes without 6-tunnel, and the standard planes are the finest 6-connected without tunnel.

# Combinatorial property of naive planes: $(m, n)$-pieces

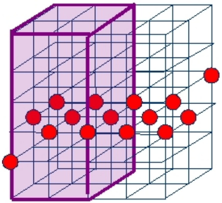
We consider the naive planes in the case of $0 \leq a \leq b \leq c$ and $c \neq 0$.
Let $m$ and $n$ be two positive integers such that $m, n \leq c$.

**Property (Reveillès, 1995)**

*In a naive plane, there are at most $mn$ combinatorially different pieces that are projected as rectangles of size $m \times n$ on the xy-plane.*




$(m, n)$-*piece*



$(3, 3)$-*piece*

# Combinatorial property of naive planes: periodicity

We consider the naive planes in the case of $0 \leq a \leq b \leq c$ and $c \neq 0$. Let $m$ and $n$ be two positive integers such that $m, n \leq c$.

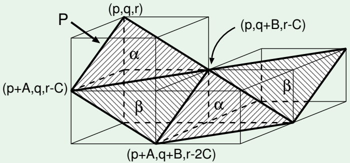
## Property (Reveillès, 1995)

*All the different configurations of $(m, n)$-pieces appear in the region that is projected on the xy-plane such as a* rectangle of size $(2n-1) \times (2m-1)$ *whose center is a leaning point.*

## Property (Kenmochi et Imiya, 2000)

*In a naive plane, there are two types of triangular pieces ($\alpha$ and $\beta$ in the figure) such that*

$$A : B : C = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$$

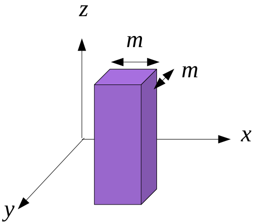


# Normal vectors for the $(m, m)$-pieces

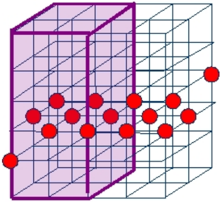
We consider the naive planes in the case of $0 \leq a \leq b \leq c$ and $c \neq 0$.
Let $m$ and $n$ be two positive integers such that $m, n \leq c$.

## Property (Vittone, 1999; Buzer ,2006)

*For every naive plane of normal vector $(a, b, c)$, the possible $(m, m)$-pieces are obtained by the 2D Farey sequence $(\frac{a}{c}, \frac{b}{c})$ of order $2(m-1)^2$.*




$(m, m)$-*piece*



$(3, 3)$-*piece*

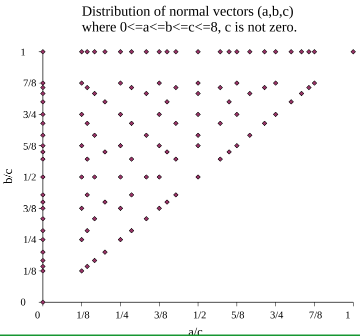
# 2D Farey sequence (Hurwitz, 1894)

## Definition (2D Farey sequence)

*The 2D Farey sequence of order n is the set of pairs of fractions:*

$$F_n = \left\{ \left( \frac{p}{q}, \frac{r}{q} \right) : gcd(p, q, r) = 1, 0 \leq p \leq q, 0 \leq r \leq q, q \leq n \right\}.$$

Example : $F_8$



# References


- I. Sivignon et I. Debled-Rennesson.
  "Droites et plans discrets," Chapitre 6 dans "Géométrie discrète et images numériques," Hermès, 2007.
- R. Klette and A. Rosenfeld.
  "2D Straightness", Chapter 9 in "Digital geometry: geometric methods for digital picture analysis," Morgan Kaufmann, 2004.
- R. Klette and A. Rosenfeld.
  "3D Straightness and Planarity," Chapter 11 in "Digital geometry: geometric methods for digital picture analysis," Morgan Kaufmann, 2004.
- V. Brimkov, D. Coeurjolly and R. Klette.
  "Digital planarity - a review," Discrete Applied Mathematics, Vol. 155, Issue 4, pp. 468–495, 2007.