Master 2 “SIS”

Digital Geometry

Topic 2:
Discrete objects and their boundaries:
adjacency graph representation

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Representation of discrete objects

- **grid point set**
- **graph** (grid points + adjacent relation)
- **complex** (grid cells + neighboring relation)

Object boundary in the Euclidean space

For $\mathbf{A} \subset \mathbb{R}^d$, the set of **interior points** is defined by

$$Int(\mathbf{A}) = \{\mathbf{x} \in \mathbf{A} : \exists r \in \mathbb{R}^+, \mathbf{U}_r(\mathbf{x}) \subseteq \mathbf{A}\}$$

where

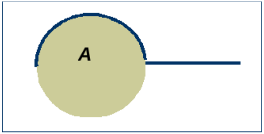
$$\mathbf{U}_r(\mathbf{x}) = \{y \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\| < r\}.$$

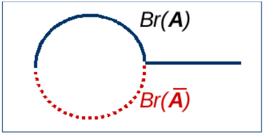
The set of **border points** is:

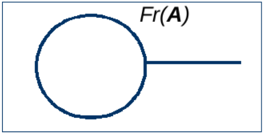
$$Br(\mathbf{A}) = \mathbf{A} \setminus Int(\mathbf{A}).$$

Then we obtain the set of **boundary points** such that

$$Fr(\mathbf{A}) = Br(\mathbf{A}) \cup Br(\overline{\mathbf{A}}) = Fr(\overline{\mathbf{A}}).$$







Object boundary in the 2D discrete space

For $\mathbf{A} \subset \mathbb{Z}^2$, the set of ***m*-interior points** is defined by

$$Int_m(\mathbf{A}) = \{\mathbf{x} \in \mathbf{A} : \mathbf{N}_m(\mathbf{x}) \subseteq \mathbf{A}\}$$

where

$$\mathbf{N}_m(\mathbf{x}) = \{y \in \mathbb{Z}^2 : \|\mathbf{x} - \mathbf{y}\| < r\}$$

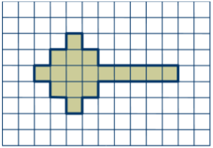
for $m = 4, 8$ if $r = 1, \sqrt{2}$ respectively.
The set of ***m*-boundary points** is:

$$Fr(\mathbf{A}) = Br_m(\mathbf{A}) \cup Br_m(\overline{\mathbf{A}})$$

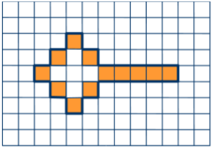
where

$$Br_m(\mathbf{A}) = \mathbf{A} \setminus Int_m(\mathbf{A}) \quad \text{m-interior border},$$
$$Br_m(\overline{\mathbf{A}}) = \overline{\mathbf{A}} \setminus Int_m(\overline{\mathbf{A}}) \quad \text{m-exterior border}.$$

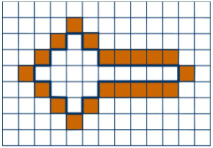
$\mathbf{A}$



$Br_4(\mathbf{A})$



$Br_4(\overline{\mathbf{A}})$



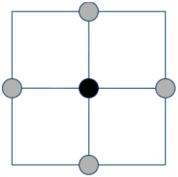
Neighborhoods in the 2D discrete space

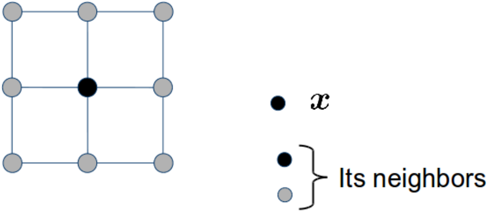
Definition (m-neighborhood)

*The m-**neighborhood** of a grid point $\mathbf{x} \in \mathbb{Z}^2$ is defined by:*

$$\mathbf{N}_m(\mathbf{x}) = \{y \in \mathbb{Z}^2 : \|\mathbf{x} - \mathbf{y}\| < r\}$$

for $m = 4, 8$ if $r = 1, \sqrt{2}$ respectively.





Object boundary in the 2D discrete space

The set of ***m*-boundary points** is:

$$Fr(\mathbf{A}) = Br_m(\mathbf{A}) \cup Br_m(\overline{\mathbf{A}})$$

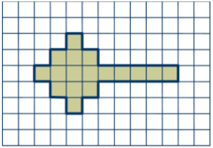
where

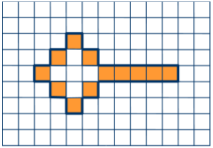
$$Br_m(\mathbf{A}) = \mathbf{A} \setminus Int_m(\mathbf{A}) \quad m\text{-interior border},$$
$$Br_m(\overline{\mathbf{A}}) = \overline{\mathbf{A}} \setminus Int_m(\overline{\mathbf{A}}) \quad m\text{-exterior border}.$$

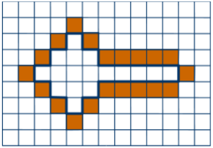
In the discrete space, a set $\mathbf{A}$ and its complement $\overline{\mathbf{A}}$ do not have the common boundary. The boundary of $\mathbf{A}$ consists of elements in $\mathbf{A}$, and that of $\overline{\mathbf{A}}$ consists of elements in $\overline{\mathbf{A}}$. (Clifford, 1956)

Alternative definition of ***m*-border points**:

$$Br_m(\mathbf{A}) = \{\mathbf{x} \in \mathbf{A} : \mathbf{N}_m(\mathbf{x}) \cap \overline{\mathbf{A}} \neq \emptyset\}.$$

$\mathbf{A}$ 

$Br_4(\mathbf{A})$ 

$Br_4(\overline{\mathbf{A}})$ 

2D Adjacency graph

Definition (m-adjancency)

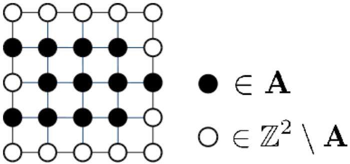
If a grid point $\mathbf{x}$ *is m-neighboring from another distinct grid point* $\mathbf{y}$, $\mathbf{x}$ *and* $\mathbf{y}$ *are* ***m*-adjacent**, *denoted by* $\mathbf{x} \in A_m(\mathbf{y})$ *and* $\mathbf{y} \in A_m(\mathbf{x})$.

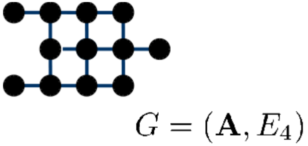
Definition (Adjacency graph (Rosenfeld 1970))

For a given grid point set $\mathbf{X} \subset \mathbb{Z}^2$, *the* **adjacency graph** *is defined by*

$$G = (\mathbf{X}, E_m)$$

where $E_m = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X} : \mathbf{y} \in A_m(\mathbf{x})\}$ *for* $m = 4, 8$.

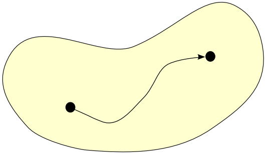




Path

Definition (m-Path)

Let X be a set of grid points. An m-path in X joining two points $\mathbf{p}$ and $\mathbf{q}$ of X is a sequence $\pi = (\mathbf{p}_0, \ldots, \mathbf{p}_n)$ of points in X such that $\mathbf{p}_0 = \mathbf{p}$, $\mathbf{p}_n = \mathbf{q}$ and $\mathbf{p}_i \in A_m(\mathbf{p}_{i-1})$ for $i = 1, \ldots, n$.

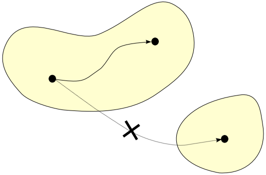


In general, $m = 4, 8$ for 2D.

Discrete object (connected component)

Definition (m-object)

A set X of grid points is an m-object if there exists an m-path in X for every pair $\mathbf{p}$ *and* $\mathbf{q}$ *of X.*



In other words, an m-object is a *connected component* of a graph $G = (X, E_m)$.

Connected component labeling (of a graph)

Algorithm (Connected components)

Input: *Graph G, starting vertex s*

- *Put s in the queue (or stack) L.*
- **while** $L \neq \emptyset$ **do**
  - *pull s from L.*
  - *Label all the neighbors of s that are not labelled and put them in L.*

It allows to calculate the connected components of a graph in **linear time**.

- **breadth-first search**
- **depth-first search**

(Hopcropft and Tarjan, 1973)

Discrete curve

Definition (closed *m*-curve)

An m-path π is a closed m-curve if every element of π has exactly two m-neighboring points in π.

Definition (*m*-curve)

An m-path π is an m-curve if for all the elements $\mathbf{p}_i$ of π, $i = 1, \ldots, n$, $\mathbf{p}_i$ has exactly two m-neighboring points in π, except for $\mathbf{p}_0$ and $\mathbf{p}_n$ that have only one.

Definition (simple *m*-curve)

Let π be an m-curve and I be the set of point indexes of π. Then, π is considered as a mapping $\pi : I \rightarrow \mathbb{Z}^2$ and said to be **simple** *if it is injective, i.e., if for all $i, j \in I$, we have*

$$\mathbf{p}_i = \mathbf{p}_j \Rightarrow i = j.$$

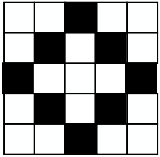
Jordan curve theorem

Theorem (Jordan curve theorem (Jordan, 1887))

Let C be a simple closed curve in the plane $\mathbb{R}^2$, called a Jordan curve. Then, its complement $\mathbb{R}^2 \setminus C$ consists of exactly two components, the interior and exterior, and C is their boundary.

Problem

The discrete version of Jordan theorem does not hold for simple closed m-curve.



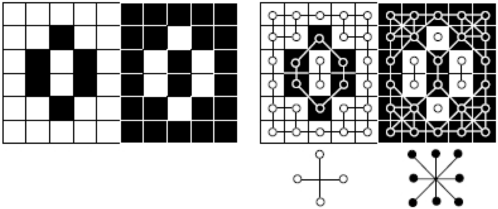
If the curve is connected, it does not disconnect its interior from its exterior (8-connectedness); if it is totally disconnected it does disconnect them (4-connectedness).

(Rosenfeld, Pflatz, 1966)

Good adjacency pairs for 2D binary images

Theorem (Separation theorem (Duda, Hart, Munson, 1967))

A simple closed m-curve C m'-separates all pixels inside C from all pixels outside C, for $(m, m') = (4, 8), (8, 4)$.



(Klette, Rosenfeld, 2003)

Definition (Generarisation: good adjacency pairs (Kong, 2001))

(α, β) is called a good pair iff, for $(m, m') \in \{(\alpha, \beta), (\beta, \alpha)\}$, any simple closed m-curve m'-separates its (at least one) m'-holes from the background and any totally m-disconnected set cannot m'-separate any m'-hole from the background.

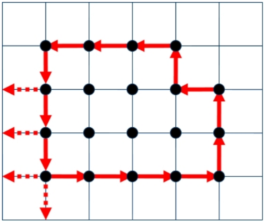
2D Border tracing

Border extraction by set operation

The complexity is linear to the image size.

Border tracing by using the *m*-neighborhood (Alexander, Thaler, 1971)

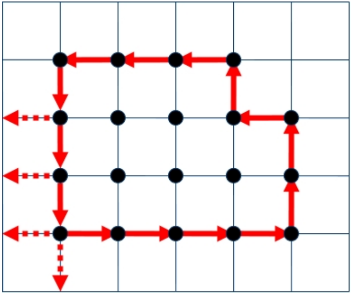
By using the cyclic order of the *m*-neighborhood, we obtain the set of border points $\partial_m \mathbf{A}$ by verifying only for the border points their neighbors.



Example: $\partial_4 \mathbf{A}$.

2D Border tracing and curve structure

The **curve structure** consisting of a sequence of grid points each of which has two neighbors is used for tracing the border of an object.



Relation between the two different discrete borders

Given $\mathbf{A} \in \mathbb{Z}^2$, we have the following relation between

- the border defined by the set operation:

$$Br_m(\mathbf{A}) = \{\mathbf{x} \in \mathbf{A} : \mathbf{N}_m(\mathbf{x}) \cap \overline{\mathbf{A}} \neq \emptyset\},$$

- the border traced by the neighborhood: $\partial_{m'}\mathbf{A}$.

Relation between $Br_m(\mathbf{A})$ and $\partial_{m'}\mathbf{A}$

$$Br_m(\mathbf{A}) = \partial_{m'}\mathbf{A}$$

for $(m, m') = (4, 8), (8, 4)$.

(Rosenfeld, 1970)

Object boundary in the 3D discrete space

For $\mathbf{A} \subset \mathbb{Z}^3$, the set of ***m*-interior points** is defined by

$$Int_m(\mathbf{A}) = \{\mathbf{x} \in \mathbf{A} : \mathbf{N}_m(\mathbf{x}) \subseteq \mathbf{A}\}$$

where

$$\mathbf{N}_m(\mathbf{x}) = \{y \in \mathbb{Z}^2 : \|\mathbf{x} - \mathbf{y}\| < r\}$$

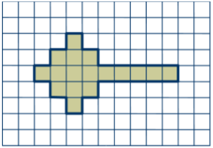
for $m = 6, 18, 26$ if $r = 1, \sqrt{2}, \sqrt{3}$ respectively.

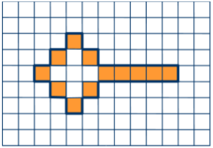
The set of ***m*-boundary points** is:

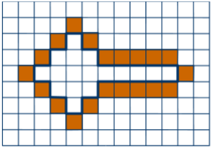
$$Fr(\mathbf{A}) = Br_m(\mathbf{A}) \cup Br_m(\overline{\mathbf{A}})$$

where

$$Br_m(\mathbf{A}) = \mathbf{A} \setminus Int_m(\mathbf{A}) \quad \text{m-interior border},$$
$$Br_m(\overline{\mathbf{A}}) = \overline{\mathbf{A}} \setminus Int_m(\overline{\mathbf{A}}) \quad \text{m-exterior border}.$$

A 

$Br_m(\mathbf{A})$ 

$Br_m(\overline{\mathbf{A}})$ 

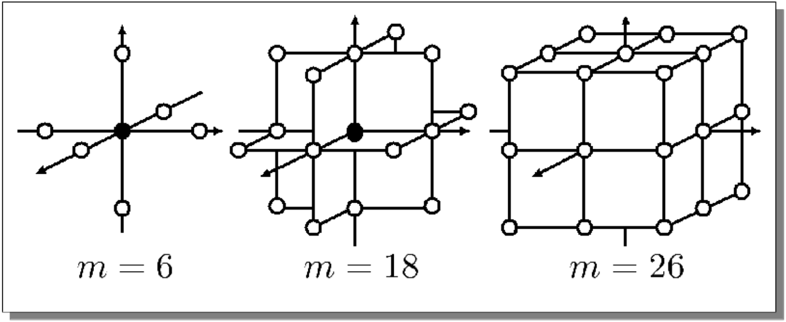
Neighborhoods in the 3D discrete space

Definition (m-neighborhood)

The ***m-neighborhood*** *of a grid point* $\mathbf{x} \in \mathbb{Z}^3$ *is defined by:*

$$\mathbf{N}_m(\mathbf{x}) = \{y \in \mathbb{Z}^3 : \|\mathbf{x} - \mathbf{y}\| < r\}$$

for $m = 6, 18, 26$ *if* $r = 1, \sqrt{2}, \sqrt{3}$ *respectively.*



3D discrete border and surface structure

Alternative definition of ***m*-border points**:

$$Br_m(\mathbf{A}) = \{\mathbf{x} \in \mathbf{A} : \mathbf{N}_m(\mathbf{x}) \cap \overline{\mathbf{A}} \neq \emptyset\}$$

for $m = 6, 18, 26$.

Question

- *How to follow interior border points?*
- *How to define a surface structure in the discrete space?*

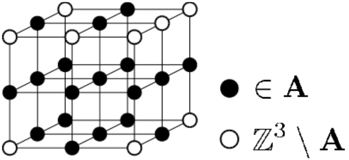
3D Adjacency graph

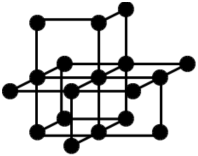
Definition (Adjacency graph (Rosenfeld 1970))

For a given grid point set $\mathbf{X} \subset \mathbb{Z}^3$, *the* **adjacency graph** *is defined by*

$$G = (\mathbf{X}, E_m)$$

where $E_m = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X} : \mathbf{y} \in A_m(\mathbf{x})\}$ *for* $m = 6, 18, 26$.





$G = (\mathbf{A}, E_6)$

Inter-voxel boundary of a discrete object

Let us consider a **discrete space** as a pair (V, W) where V is a countable set and W is a symmetric relation on $V \times V$.

For example: $(V, W) = (\mathbb{Z}^2, 4), (\mathbb{Z}^3, 6)$.

Definition (Inter-voxel (pixel) boundary)

Let (V, W) be a discrete space, and $\mathbf{X}$ be a subset of V. The **boundary** *of $\mathbf{X}$ and its complement $\overline{\mathbf{X}}$ is defined by*

$$\partial(\mathbf{X}, \overline{\mathbf{X}}) = \{(\mathbf{u}, \mathbf{v}) \in W : \mathbf{u} \in \mathbf{X} \wedge \mathbf{v} \in \overline{\mathbf{X}}\}.$$

Note that every element of $\partial(\mathbf{X}, \overline{\mathbf{X}})$ is directed.

Inter-voxel surface

Definition (Inter-voxel surface)

Given a discrete space (V, W), a discrete surface S is defined as a non-empty subset of W.

Then, we have

- the immediate interior $II(S) = \{u : (u, v) \in S \text{ for } v \in V\}$,
- the immediate exterior $IE(S) = \{v : (u, v) \in S \text{ for } u \in V\}$.

Definition (Almost-Jordan discrete surface)

Given a discrete space (V, W), a discrete surface S is almost-Jordan iff every W-path from an element of $II(S)$ to an element of $IE(S)$ crosses S.

$\kappa\lambda$-Jordan discrete surface theorem

Definition ($\kappa\lambda$-Jordan discrete surface)

A discrete surface S is $\kappa\lambda$-Jordan iff it is almost-Jordan, its interior is κ-connected, and its exterior is λ-connected.

Theorem ($\kappa\lambda$-Jordan discrete surface theorem (Herman, 1998))

Let P be a κ-connected subset of V and Q be a λ-connected union of W-components of the complement of P in V. Then, the boundary $S = \partial(P, Q)$ is $\kappa\lambda$-Jordan.

Examples of pairs of Jordan:

- $\{8, 4\}, \{8, 8\}$ for the discrete space $(\mathbb{Z}^2, 4)$,
- $\{18, 6\}, \{26, 6\}$ for the discrete space $(\mathbb{Z}^3, 6)$.

Inter-voxel boundary following

Algorithm: 3D boundary following (Aztzy et al., 1981)

Input: 6-object, starting 2-cell s
Output: Set F of 2-cells that form the boundary

- Put s in a list F and in a queue Q, and also twice in a list L.
- **while** $Q \neq \emptyset$ **do**
  - Pull f from Q.
  - **for each** successor neighbor g of f **do**
    - **if** g is in L, pull g from L.
    - **otherwise** put g in F, in Q and in L.

The graph structure and the similar idea to the graph traversal are used.

References

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