Master 2 “SIS”

Digital Geometry

Topic 1 :
Introduction: digital images and discretization models

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October 24, 2011

What is digital geometry?

Digital geometry

Digital geometry is a branch of computer science devoted to study geometrical (and topological) problems of digitized objects in digital images, i.e. it is **geometry for digital image analysis and synthesis**.

Since digital images (input data) are represented by integers, the computations can be made by using only **integers (exact computations)**.

Image analysis = content analysis

- color analysis
- texture analysis
- shape analysis
- ...

Why digital geometry?

Advantages

- exact computation (no computation error)
- precision analysis due to image resolution
- a finite number of discrete shapes
- efficient algorithms
- ...

Related domains to digital geometry

- discrete geometry and topology
- graph theory
- computational geometry
- combinatorial geometry
- number theory
- approximation and estimation
- mathematical morphology
- image processing and analysis
- computer graphics
- computer vision
- pattern recognition
- ...

History around digital geometry

- late 1960s: the term **pixel: picture element** was presented (JetPropulsion Laboratory, California)
- late 1970s: the birth of **digital geometry** in the united states (Azriel Rosenfeld)
- In France, the research is always very active.
  Pioneers: *Jean-Marc Chassary, Jean Françon, Jean-Pierre Reveillès, ...*

in terms of digital image "dimension"

- 1980s: work for 2D images
- 1990s: work for 3D images
- 2000s: work for nD images

Lecture schedule

24/10/2011	**Topic 1:**	Introduction: digital images and discretization models
31/10/2011	**Topic 2:**	Representations of digital objects and their boundary
7/11/2011	**Topic 3:**	Digital curves and surfaces
14/11/2011	**Topic 4:**	Geometric properties of digital objects
21/11/2011	**Topic 5:**	Digital shape recognition
28/11/2011	**Topic 6:**	Geometric transformation of digital images

Continuous and digital images

Definition (Continuous image)

An nD image is defined as a function $\mathcal{I} : \mathbb{R}^n \rightarrow V$ *where* V *is a value space containing at least two elements.*

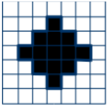
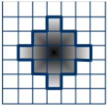
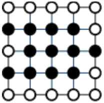
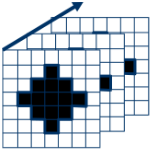
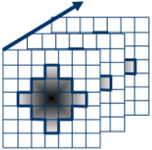
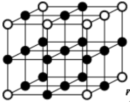
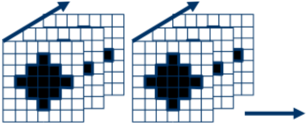
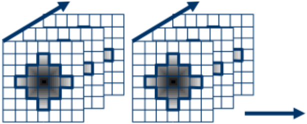
Definition (Digital image)

An nD digital image is defined as a function $I : \mathbb{Z}^n \rightarrow V$.

Examples for V:

- scalars: $\mathbb{R}$, $\mathbb{Z}$, $\{0, 1\}$, $\{x \in \mathbb{Z} : 0 \leq x \leq 255\}$, ...
- vectors: $\mathbb{R}^k$, $\mathbb{Z}^k$, ...
- tensors

Digital images and discrete spaces

dim.	binary image	gray image	discrete space
2D			 $\mathbb{Z}^2$
3D			 $\mathbb{Z}^3$
4D			

Digitization

Definition (Digitization)

Digitization consists of two parts:

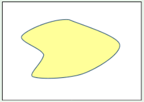
- **discretization:** *sample the value of an analog signal at regular intervals;*
- **quantization:** *round those samples to a fixed set of numbers such as integers.*

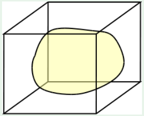
For an nD digital image, those **regular intervals** make a grid, *i.e.* the **discrete space** $\mathbb{Z}^n$.

Continuous space and discrete space

object in continuous space

$S \subset \mathbb{R}^n$

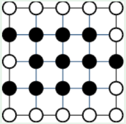


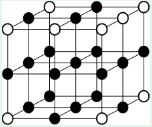


$\downarrow$ **Discretization**

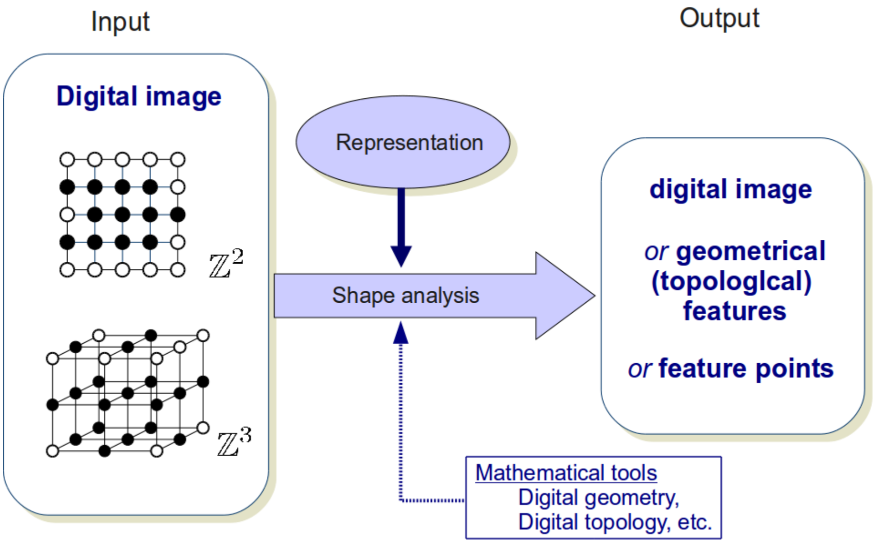
object in discrete space

$D(S) \subset \mathbb{Z}^n$





Discrete shape analysis



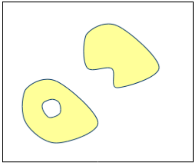
Digital geometry and topology

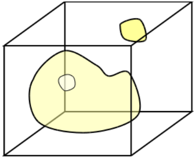
Geometrical features of discrete objects

Examples: straightness, planarity (linearity), circularity, sphericity (roundness), convexity, concavity, curvature, perimeter (length), area, volume, centroid (moments), ...

Topological features of discrete objects

Examples: object boundary, curve, surface, number of objects (of connected components), number of holes in an object, shape deformation preserving the topology, ...





Discretization and shape analysis

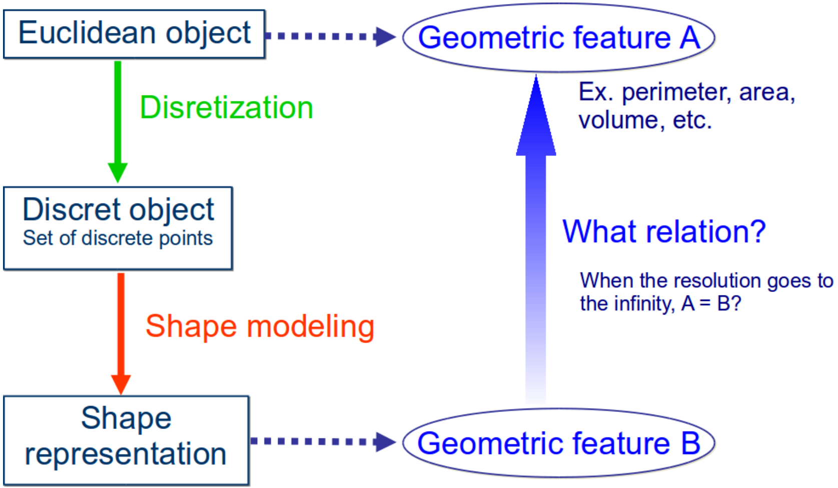
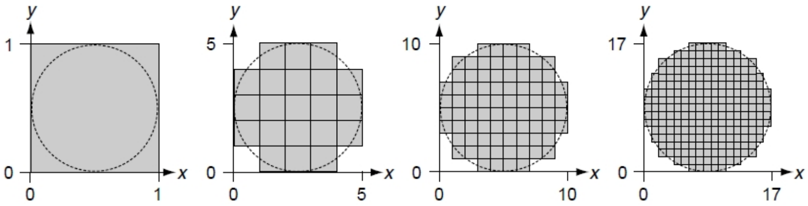


Image resolution and discrete space

Image (grid) resolution is the inverse of the grid interval, which is generally set to be 1.

Definition (Multigrid discrete space)

Let $h > 0$ be a grid resolution and $\mathbb{Z}_h = \{i/h : i \in \mathbb{Z}\}$; then $\mathbb{Z}_h^n$ is the set of nD discrete points in a grid of resolution of h.



(Klette and Rosenfeld, 2003)

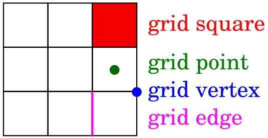
Grid points and grid cells

Definition (Grid points)

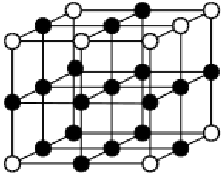
The grid point set is $\mathbb{Z}^n$ ($\mathbb{Z}_h^n$).

Definition (Grid cells)

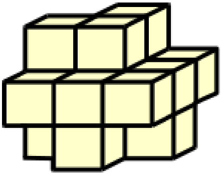
A grid vertex is also called a 0-cell; a grid edge is a 1-cell; a grid square is a 2-cell; a grid cube is called a 3-cell;.....



2D grid



3D grid point model



3D grid cell model

Properties expected to be preserved by discretization

symmetry

If $\mathbf{S}$ is symmetric, its discretization $D(\mathbf{S})$ is symmetric in the same way.

connectedness

If $\mathbf{S}$ is connected, $D(\mathbf{S})$ is connected as well.

topology

If $\mathbf{S}$ is a curve (surface), $D(\mathbf{S})$ is a curve (surface) as well.

multigrid convergence

When the image resolution h goes to the infinity, the series of $D_h(\mathbf{S})$ converges such that their limit is equal to $\mathbf{S}$.

Discretization models

Let $S \subset \mathbb{R}^n$ be an original object.

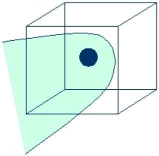
Definition (Gauss discretization)

The **Gauss discretization** $G(\mathbf{S})$ *is the union of the n-cells with center points in S.*

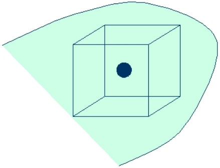
Definition (Jordan discretization)

The **inner Jordan discretization** $J^{-}(\mathbf{S})$ *is the union of n-cells that are completely contained in S. The* **outer Jordan discretization** $J^{+}(\mathbf{S})$ *is the union of all such n-cells that have nonempty intersections with S.*

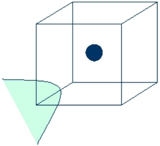
The outer Jordan discretization is also called **super-cover discretization**.



$G(\mathbf{S})$

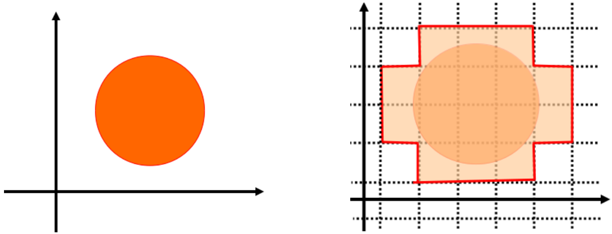


$J^{-}(\mathbf{S})$



$J^{+}(\mathbf{S})$

Example of super-cover discretization: disk



For $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{c}\| \leq r\}$,

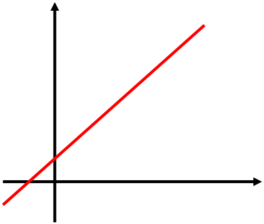
$$D(\mathbf{S}) = \{\mathbf{p} \in \mathbb{Z}^2 : Cell(\mathbf{p}) \cap \mathbf{S} \neq \emptyset\}$$

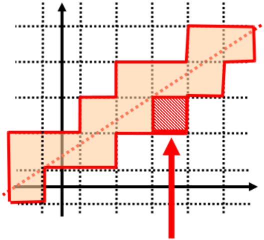
where $Cell(\mathbf{p}) = \left[p_x - \frac{1}{2}, p_x + \frac{1}{2}\right] \times \left[p_y - \frac{1}{2}, p_y + \frac{1}{2}\right]$.

Properties of super-cover discretization

- $D(\mathbf{X} \cup \mathbf{Y}) = D(\mathbf{X}) \cup D(\mathbf{Y})$,
- $D(\mathbf{X}) = \cup_{\mathbf{p} \in \mathbf{X}} D(\{\mathbf{p}\})$,
- $D(\mathbf{X} \cap \mathbf{Y}) \subseteq D(\mathbf{X}) \cap D(\mathbf{Y})$,
- if $\mathbf{X} \subseteq \mathbf{Y}$, then $D(\mathbf{X}) \subseteq D(\mathbf{Y})$.

Example of super-cover discretization: straight line





Topology is not always preserved!

Solution

Minimal-cover

Symmetry is not always preserved.

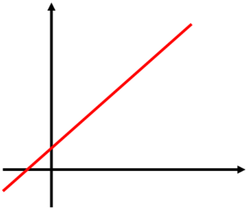
Grid-intersection discretization

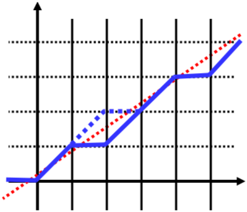
Gauss and inner Jordan discretization are not appropriate for curves or arcs.

Definition (Grid-intersection discretization)

Let $\gamma \subset \mathbb{R}^2$ be a curve. The **grid-intersection discretization** $R(\gamma)$ *is the set of all grid points that are closest (in Euclidean distance) to the intersections points of γ with the grid lines.*

This can be extended to 3D curves or arcs.

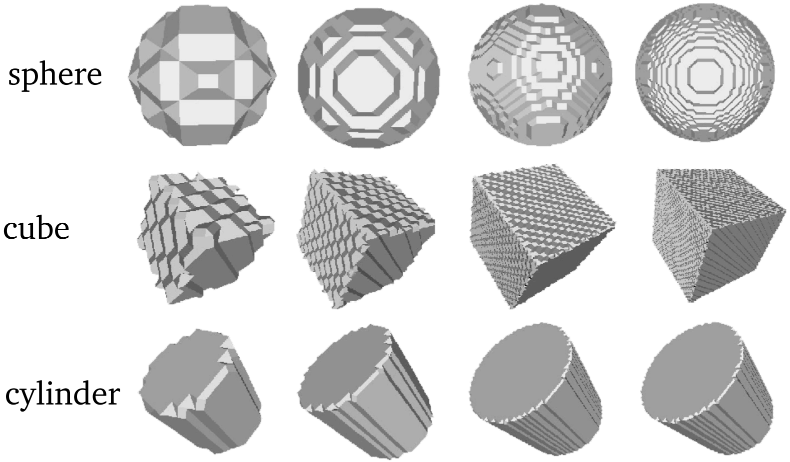




When two grid points are in the same distance from the intersection, the choice is needed.

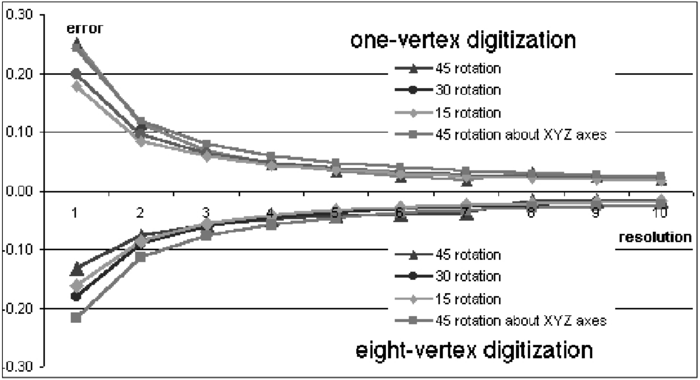
Thus, symmetry is not always preserved.

Multigrid discrete objects



Volume computation and multigrid convergence

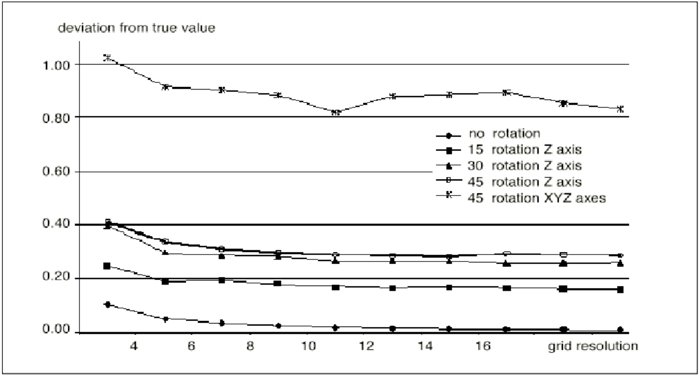
Volumes for a cube with different viewpoints are computed in different grid resolutions.



Local volume elements converge towards the true volume value (known since end of 19th century: C. Jordan, G. Peano et al.).

Surface area computation and multigrid convergence

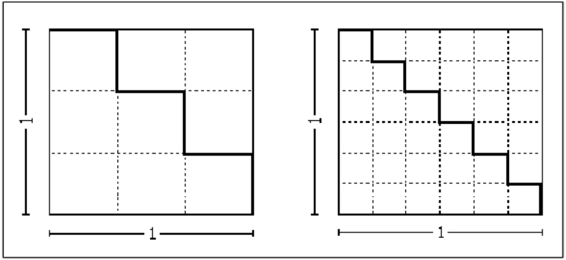
Surface areas for a cube with different viewpoints are computed in different grid resolutions.

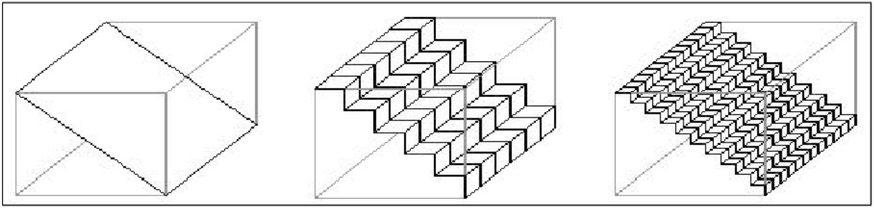


Problem

Different surface area estimates for one cube from different perspectives (using marching cubes algorithms).

Multigrid convergence for perimeter and surface area





How to obtain correct geometrical features from discrete objects?

Principal idea is to add some **hypothesis on the shape**: for example,

- dimension of object,
- number of objects,
- number of holes,
- smoothness (of curves and surfaces),
- linearity,
- circularity,
- ...

You need to know what you would like to see.

References

- Reinhard Klette and Azriel Rosenfeld.
  "Digital geometry: geometric methods for digital picture analysis",
  San Diego: Morgan Kaufmann, 2004.
- David Coeurjolly, Annick Montanvert, et Jean-Marc Chassery.
  "Géométrie discrète et images numériques", Lavoisier, 2007.

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