Some morphological operators in graph spaces

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Abstract. We study some basic morphological operators acting on the lattice of all subgraphs of a (non-weighted) graph $\mathbb{G}$. To this end, we consider two dual adjunctions between the edge set and the vertex set of $\mathbb{G}$. This allows us (i) to recover the classical notion of a dilation/erosion of a subset of the vertices of $\mathbb{G}$ and (ii) to extend it to subgraphs of $\mathbb{G}$. Afterward, we propose several new erosions, dilations, granulometries and alternate filters acting (i) on the subsets of the edge and vertex set of $\mathbb{G}$ and (ii) on the subgraphs of $\mathbb{G}$.

1 Introduction

From a formal point of view, digital image processing historically consists of analyzing the transformations that act on the subsets of $\mathbb{Z}^2$ (the sets of pixels in a binary image) and the transformations that act on the maps from $\mathbb{Z}^2$ to $\mathbb{N}$ (the images themselves). In such a perspective, mathematical morphology provides a set of filtering and segmenting tools that are very useful in applications.

On the other hand, there is a growing interest for considering digital objects not only composed of points but also composed of elements lying between them and carrying structural information about how the points are glued together (see [1, 2] for recent examples). The simplest of these representations are the (non-weighted) graphs. The domain of an image is considered as a graph whose vertex set is made of the pixels and whose edge set is given by an adjacency relation on these pixels. In this context, it becomes relevant to consider the transformations acting on the set of all subgraphs and not only those acting on the set of all subsets of pixels.

When dealing with a graph $\mathbb{G}$, we often need (see *e.g.* ([1, 3–5]) to consider the graph induced by a subset S of vertices of $\mathbb{G}$. To this end, we associate with S the largest subset of edges of $\mathbb{G}$ such that the obtained pair is a graph. In other cases, we have to consider a graph induced by a subset of the edges of $\mathbb{G}$.

Motivated by classifying and understanding these operations and their combinations, we propose a systematic study of the basic operators which are used to derive a set of edges from a set of vertices and a set of vertices from a set of edges. It turns out that these operators are dilations and erosions. They allow us (i) to recover the classical notion of a dilation/erosion of a subset of vertices and (ii) to extend it to subgraphs (Section 3). Then, we propose several new erosions, dilations, granulometries and alternate sequential filters acting (i) on the subsets of edges and on the subsets of vertices and (ii) on the subgraphs. We emphasize that, contrarily to most of the previous work on morphology in graphs (such as [6–9]), the main operators of this paper input and output graphs.

The proofs of the properties presented in this paper will be given in a future extended version [10].

2 Lattice of graphs

We define a *graph* as a pair $X = (X^\bullet, X^\times)$ where $X^\bullet$ is a set and $X^\times$ is composed of unordered pairs of distinct elements in $X^\bullet$, *i.e.*, $X^\times$ is a subset of $\{\{x, y\} \subseteq X^\bullet \mid x \neq y\}$. Each element of $X^\bullet$ is called a *vertex or a point (of X)*, and each element of $X^\times$ is called an *edge (of X)*. In the sequel, to simplify the notations, $e_{x,y}$ stands for the edge $\{x, y\} \in X^\times$.

Let X and Y be two graphs. If $Y^\bullet \subseteq X^\bullet$ and $Y^\times \subseteq X^\times$, then X and Y are ordered and we write $Y \sqsubseteq X$. If $Y \sqsubseteq X$, we say that Y is a *subgraph* of X, or that Y is *smaller* than X and that X is *greater* than Y.

Important remark. Hereafter, the workspace is a graph $\mathbb{G} = (\mathbb{G}^\bullet, \mathbb{G}^\times)$ and we consider the sets $\mathcal{G}^\bullet$, $\mathcal{G}^\times$ and $\mathcal{G}$ of respectively all subsets of $\mathbb{G}^\bullet$, all subsets of $\mathbb{G}^\times$ and all subgraphs of $\mathbb{G}$.

Let $\mathcal{S}_0, \mathcal{S}_1 \subseteq \mathcal{G}$ be the sets of respectively the graphs made of a single vertex and the graphs made of a pair of vertices linked by an edge, *i.e.*, $\mathcal{S}_0 = \{(\{x\}, \emptyset) \mid x \in \mathbb{G}^\bullet\}$ and $\mathcal{S}_1 = \{(\{x, y\}, \{e_{x,y}\}) \mid e_{x,y} \in \mathbb{G}^\times\}$. We set $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. Any graph $X \in \mathcal{G}$, is *generated* by the family $\mathcal{F} = \{X_1, \ldots, X_\ell\}$ of all elements in $\mathcal{S}$ smaller than X: $X = (\bigcup_{i\in[1,\ell]} {X_i}^\bullet, \bigcup_{i\in[1,\ell]} {X_i}^\times)$; we say that the elements of $\mathcal{F}$ are the *generators* of X. Conversely, any family $\mathcal{F}$ of elements in $\mathcal{S}$ generates an element of $\mathcal{G}$. Hence, $\mathcal{S}$ *(sup-) generates* $\mathcal{G}$.

Clearly, the ordering $\sqsubseteq$ on graphs amount to say that $Y \sqsubseteq X$ when all generators of Y are also generators of X. Therefore, ordering $\sqsubseteq$ provides a *lattice* structure on the set $\mathcal{G}$. Indeed, the largest graph smaller than a family $\mathcal{F} = \{X_1, \ldots, X_\ell\}$ of elements in $\mathcal{G}$ is the graph generated by the generators common to all X_i, $i \in [1, \ell]$; this *infimum* is denoted by $\sqcap\mathcal{F}$. Similarly, the *supremum* $\sqcup\mathcal{F}$ is generated by the union of the families of generators of all X_i, $i \in [1, \ell]$.

If $X^\bullet \subseteq \mathbb{G}^\bullet$ (resp. $Y^\times \subseteq \mathbb{G}^\times$), we denote by $\overline{X^\bullet}$ (resp. $\overline{Y^\times}$) the *complementary set of* $X^\bullet$ *(resp.* $Y^\times$*) in* $\mathbb{G}^\bullet$ *(resp.* $\mathbb{G}^\times$*)*, that is $\overline{X^\bullet} = \mathbb{G}^\bullet \setminus X^\bullet$ (resp. $\overline{Y^\times} = \mathbb{G}^\times \setminus Y^\times$). Observe that, if X is a subgraph of $\mathbb{G}$, then, except in some degenerated cases, the pair $(\overline{X^\bullet}, \overline{X^\times})$ is not a graph.

Property 1 *The set* $\mathcal{G}$ *of the subgraphs of* $\mathbb{G}$ *form a complete lattice, sup-generated by the set* $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$*, but not complemented. The supremum and the infimum of any family* $\mathcal{F} = \{X_1, \ldots X_\ell\}$ *of elements in* $\mathcal{G}$ *are given by respectively* $\sqcap\mathcal{F} = (\bigcap_{i\in[1,\ell]} {X_i}^\bullet, \bigcap_{i\in[1,\ell]} {X_i}^\times)$ *and* $\sqcup\mathcal{F} = (\bigcup_{i\in[1,\ell]} {X_i}^\bullet, \bigcup_{i\in[1,\ell]} {X_i}^\times)$.

3 Dilations and erosions

In the graph $\mathbb{G}$, we can consider sets of points as well as sets of edges. Therefore, it is convenient to consider operators to go from one kind of sets to the other one.

In this section, we investigate such operators and we study their morphological properties. Then, based on these operators, we propose several dilations and erosions acting on the lattice of all subgraphs of $\mathbb{G}$.

Let $X^\bullet$ be a subset of $\mathbb{G}^\bullet$, we denote by $\mathcal{G}_{X^\bullet}$ the set of all subgraphs of $\mathbb{G}$ whose vertex set is $X^\bullet$. Let $Y^\times$ be a subset of $\mathbb{G}^\times$. We denote by $\mathcal{G}_{Y^\times}$ the set of all subgraphs of $\mathbb{G}$ whose edge set is $Y^\times$.

Definition 2 (edge-vertex correspondences) *We define the operators $\delta^\bullet$, $\epsilon^\bullet$ from $\mathcal{G}^\times$ into $\mathcal{G}^\bullet$ and the operators $\epsilon^\times, \delta^\times$ from $\mathcal{G}^\bullet$ into $\mathcal{G}^\times$ as follows:*

	$\mathcal{G}^\times \to \mathcal{G}^\bullet$	$\mathcal{G}^\bullet \to \mathcal{G}^\times$
Provide the object with a graph structure	$X^\times \to \delta^\bullet(X^\times)$ *such that* $(\delta^\bullet(X^\times), X^\times) = \sqcap \mathcal{G}_{X^\times}$	$X^\bullet \to \epsilon^\times(X^\bullet)$ *such that* $(X^\bullet, \epsilon^\times(X^\bullet)) = \sqcup \mathcal{G}_{X^\bullet}$
Provide its complement with a graph structure	$X^\times \to \epsilon^\bullet(X^\times)$ *such that* $(\overline{\epsilon^\bullet(X^\times)}, \overline{X^\times}) = \sqcap \mathcal{G}_{\overline{X^\times}}$	$X^\bullet \to \delta^\times(X^\bullet)$ *such that* $(\overline{X^\bullet}, \overline{\delta^\times(X^\bullet)}) = \sqcup \mathcal{G}_{\overline{X^\bullet}}$

In other words, if $X^\bullet \subseteq \mathbb{G}^\bullet$ and $Y^\times \subseteq \mathbb{G}^\times$, $(\delta^\bullet(Y^\times), Y^\times)$ is the smallest subgraph of $\mathbb{G}$ whose edge set is $Y^\times$, $(X^\bullet, \epsilon^\times(X^\bullet))$ is the largest subgraph of $\mathbb{G}$ whose vertex set is $X^\bullet$, $(\overline{\epsilon^\bullet(Y^\times)}, \overline{Y^\times})$ is the smallest subgraph of $\mathbb{G}$ whose edge set is $\overline{Y^\times}$, and $(\overline{X^\bullet}, \overline{\delta^\times(X^\bullet)})$ is the largest subgraph of $\mathbb{G}$ whose vertex set is $\overline{X^\bullet}$.

These operators are illustrated in Figs. 1a-f. The following property locally characterize them. This property leads in particular to simple linear-time algorithms (with respect to the cardinality of $\mathbb{G}^\bullet$ and $\mathbb{G}^\times$) to compute $\delta^\bullet(X^\times)$, $\epsilon^\times(Y^\bullet)$, $\epsilon^\bullet(X^\times)$ and $\epsilon^\bullet(X^\times)$ without explicitly considering the families $\mathcal{G}_{X^\times}$, $\mathcal{G}_{X^\bullet}$, $\mathcal{G}_{\overline{X^\times}}$ and $\mathcal{G}_{\overline{X^\bullet}}$.

Property 3 *For any $X^\times \subseteq \mathbb{G}^\times$ and $Y^\bullet \subseteq \mathbb{G}^\bullet$:*

1. *$\delta^\bullet : \mathbb{G}^\times \to \mathbb{G}^\bullet$ is such that $\delta^\bullet(X^\times) = \{x \in \mathbb{G}^\bullet \mid \exists e_{x,y} \in X^\times\}$;*
2. *$\epsilon^\times : \mathbb{G}^\bullet \to \mathbb{G}^\times$ is such that $\epsilon^\times(Y^\bullet) = \{e_{x,y} \in \mathbb{G}^\times \mid x \in Y^\bullet$ and $y \in Y^\bullet\}$;*
3. *$\epsilon^\bullet : \mathbb{G}^\times \to \mathbb{G}^\bullet$ is such that $\epsilon^\bullet(X^\times) = \{x \in \mathbb{G}^\bullet \mid \forall e_{x,y} \in \mathbb{G}^\times, e_{x,y} \in X^\times\}$;*
4. *$\delta^\times : \mathbb{G}^\bullet \to \mathbb{G}^\times$ is such that $\delta^\times(Y^\bullet) = \{e_{x,y} \in \mathbb{G}^\times \mid$ either $x \in Y^\bullet$ or $y \in Y^\bullet\}$.*

In other words, $\delta^\bullet(X^\times)$ is the set of all vertices which belong to an edge of $X^\times$, $\epsilon^\times(Y^\bullet)$ is the set of all edges whose two extremities are in $Y^\bullet$, $\epsilon^\bullet(X^\times)$ is the set of all vertices which do not belong to any edge of $\overline{X^\times}$, and $\delta^\times(Y^\bullet)$ is the set of all edges which have at least one extremity in $Y^\bullet$.

From this characterization, we can recognize the general graph version of some operators introduced by Meyer and Angulo [8] (see also [9]) for the hexagonal grid. However, unlike Property 3, the important theorem of structure 9 does not appear in [8] or [9], neither Theorem 12 nor Property 14.

Before further analyzing the operators defined above, let us briefly recall some algebraic tools which are fundamental in mathematical morphology [11].

Given two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, an operator $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ is called a *dilation* when it preserves the supremum (*i.e.* $\forall \mathcal{E} \subseteq \mathcal{L}_1, \delta(\vee_1 \mathcal{E}) = \vee_2\{\delta(X) \mid X \in \mathcal{E}\}$, where $\vee_1$ is the supremum in $\mathcal{L}_1$ and $\vee_2$ the supremum in $\mathcal{L}_2$). Similarly, an operator which preserves the infimum is called an *erosion*.

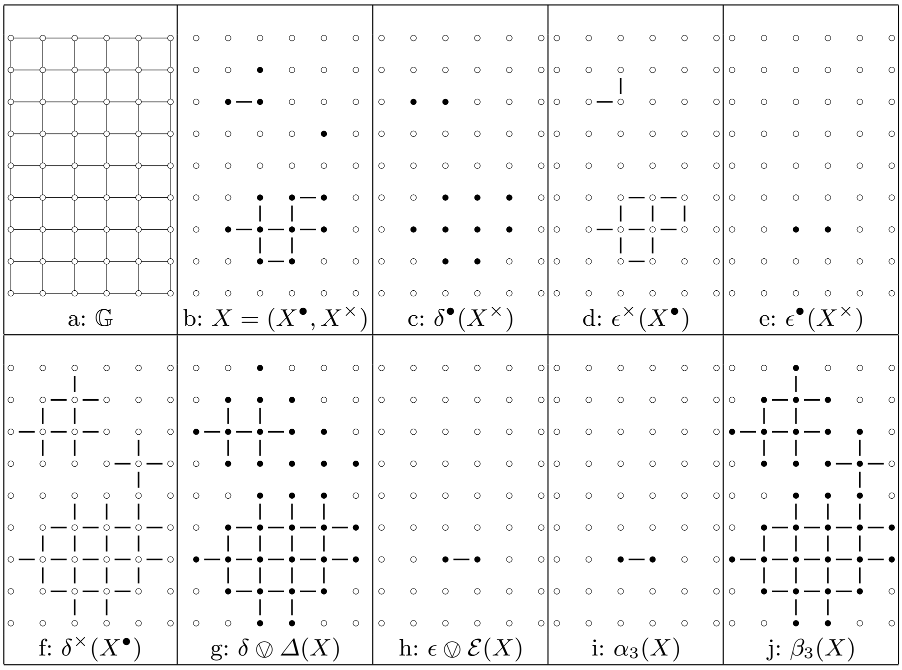

Fig. 1. Dilations and erosions.

Two operators $\epsilon : \mathcal{L}_1 \to \mathcal{L}_2$ and $\delta : \mathcal{L}_2 \to \mathcal{L}_1$ form an *adjunction* (ϵ, δ) when for any X in $\mathcal{L}_2$ and any Y in $\mathcal{L}_1$, we have $\delta(X) \leq_1 Y \Leftrightarrow X \leq_2 \epsilon(Y)$, where $\leq_1$ and $\leq_2$ denote the order relations on respectively $\mathcal{L}_1$ and $\mathcal{L}_2$. Given two operators ϵ and δ, if the pair (ϵ, δ) is an adjunction, then ϵ is an erosion and δ is a dilation.

Given two complemented lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, two operators α and β from $\mathcal{L}_1$ into $\mathcal{L}_2$ are *dual (with respect to the complement) of each other* when, for any $X \in \mathcal{L}_1$, we have $\beta(X) = \overline{\alpha(\overline{X})}$. If α and β are dual of each other, then β is an erosion whenever α is a dilation.

Property 4 (dilation, erosion, adjunction, duality)

1. *Both* $(\epsilon^\times, \delta^\bullet)$ *and* $(\epsilon^\bullet, \delta^\times)$ *are adjunctions.*
2. *Operators* $\epsilon^\times$ *and* $\delta^\times$ *(resp.* $\epsilon^\bullet$ *and* $\delta^\bullet$*) are dual of each other.*
3. *Operators* $\delta^\bullet$ *and* $\delta^\times$ *are dilations.*
4. *Operators* $\epsilon^\bullet$ *and* $\epsilon^\times$ *are erosions.*

Let us compose these dilations and erosions to act on $\mathcal{G}^\bullet$ and $\mathcal{G}^\times$.

Definition 5 (vertex-dilation, vertex-erosion) *We define* δ *and* ϵ *that act on* $\mathcal{G}^\bullet$ (i.e., $\mathcal{G}^\bullet \to \mathcal{G}^\bullet$*) by* $\delta = \delta^\bullet \circ \delta^\times$ *and* $\epsilon = \epsilon^\bullet \circ \epsilon^\times$.

As compositions of respectively dilations and erosions, δ and ϵ are respectively a dilation and an erosion. Moreover, by composition of adjunctions and dual operators, δ and ϵ are dual and (ϵ, δ) is an adjunction.

In fact, it can be shown that δ and ϵ correspond exactly to the usual notions of an erosion and of a dilation of a set of vertices in a graph [6]. It means, in particular that, when $\mathbb{G}^\bullet$ is a subset of the grid points $\mathbb{Z}^d$ and when the edge set $\mathbb{G}^\times$ is obtained from a symmetrical structuring element, then the operators defined above are equivalent to the usual binary dilation and erosion by the considered structuring element. For instance, in Fig. 1, $\mathbb{G}^\bullet$ is a rectangular subset of $\mathbb{Z}^2$ and $\mathbb{G}^\times$ corresponds to the basic "cross" structuring element. It can be verified that the vertex sets in Fig. 1g and h, obtained by applying δ and ϵ to $X^\bullet$ (Fig. 1b), are the dilation and the erosion by a "cross" structuring element of $X^\bullet$.

We now consider a dual/adjunct pair of dilation and erosion acting on $\mathcal{G}^\times$.

Definition 6 (edge-dilation, edge-erosion) *We define Δ and $\mathcal{E}$ that act on $\mathcal{G}^\times$ by $\Delta = \delta^\times \circ \delta^\bullet$ and $\mathcal{E} = \epsilon^\times \circ \epsilon^\bullet$.*

Definition 7 *We define the operators $\delta \circledvee \Delta$ and $\epsilon \circledvee \mathcal{E}$ by respectively $(\delta(X^\bullet), \Delta X^\times)$ and $(\epsilon(X^\bullet), \mathcal{E}(X^\times))$, for any $X \in \mathcal{G}$.*

For instance, Figs. 1f and 1g present the results obtained by applying the operator $\delta \circledvee \Delta$ and the operator $\epsilon \circledvee \mathcal{E}$ to the subgraph X (Fig. 1b) of $\mathbb{G}$ (Fig. 1a).

Lemma 8 *The family $\mathcal{G}$ is closed under the operators $\delta \circledvee \Delta$ and $\epsilon \circledvee \mathcal{E}$. More precisely, for any subgraph X of $\mathbb{G}$, both $\delta \circledvee \Delta(X)$ and $\epsilon \circledvee \mathcal{E}(X)$ are subgraphs of $\mathbb{G}$.*

Theorem 9 (graph-dilation, graph-erosion) *The operators $\delta \circledvee \Delta$ and $\epsilon \circledvee \mathcal{E}$ are respectively a dilation and an erosion acting on the lattice $(\mathcal{G}, \sqsubseteq)$. Furthermore, $(\epsilon \circledvee \mathcal{E}, \delta \circledvee \Delta)$ is an adjunction.*

Note that since lattice $\mathcal{G}$ is sup-generated by set $\mathcal{S}$, it suffices to know the dilation of the graphs in $\mathcal{S}$ for characterizing the dilation of the graphs in $\mathcal{G}$.

Compared to classical morphological operators on sets, the dilations and erosions introduced in this section furthermore convey some connectivity properties different than the ones which can be deduced from classical dilations and erosions. Observe, for instance, in Fig. 1g, that some 4-adjacent vertices of $\delta(X^\bullet)$ are not linked by an edge in the graph $\delta \circledvee \Delta(X)$. These properties can be useful in further processing involving for instance connected operators [12–15].

Thanks to the operators presented in Definition 2, other intersecting adjunctions (hence dilations/erosions) can be defined on $\mathcal{G}$:

1. (α_1, β_1) such that $\forall X \in \mathcal{G}$, $\alpha_1(X) = (\mathbb{G}^\bullet, X^\times)$ and $\beta_1(X) = (\delta^\bullet(X^\times), X^\times)$;
2. (α_2, β_2) such that $\forall X \in \mathcal{G}$, $\alpha_2(X) = (X^\bullet, \epsilon^\times(X^\bullet))$ and $\beta_2(X) = (X^\bullet, \emptyset)$;
3. (α_3, β_3) such that $\forall X \in \mathcal{G}$, $\alpha_3(X) = (\epsilon^\bullet(X^\times), \epsilon^\times \circ \epsilon^\bullet(X^\times))$ and $\beta_3(X) = (\delta^\bullet \circ \delta^\times(X^\bullet), \delta^\times(X^\bullet))$.

The adjunction (α_3, β_3) is illustrated in Fig. 1i and 1j. Note also that, using usual graph terminologies, β_1 (resp. α_2) can be defined as the operator which associates to a graph the graph induced by its edge set (resp. vertex set).

4 Filters

In mathematical morphology, a *filter* is an operator α acting on a lattice $\mathcal{L}$, which is increasing (*i.e.* $\forall X, Y \in \mathcal{L}$, $\alpha(X) \leq \alpha(Y)$ whenever $X \leq Y$) and idempotent (*i.e.* $\forall X \in \mathcal{L}$, $\alpha(\alpha(X)) = \alpha(X)$). A filter α on $\mathcal{L}$ which is extensive (*i.e.* $\forall X \in \mathcal{L}$, $X \leq \alpha(X)$) is called a *closing on* $\mathcal{L}$ whereas a filter α on $\mathcal{L}$ which is anti-extensive (*i.e.* $\forall X \in \mathcal{L}$, $\alpha(X) \leq X$) is called an *opening on* $\mathcal{L}$. It is known that composing the two operators of an adjunction yields an opening or a closing depending on the order in which the operators are composed [11]. In this section, the operators of Section 3 are composed to obtain filters on $\mathcal{G}^\bullet$, $\mathcal{G}^\times$ and $\mathcal{G}$.

Definition 10 (opening, closing)

1. *We define* γ_1 *and* ϕ_1*, that act on* $\mathcal{G}^\bullet$*, by* $\gamma_1 = \delta \circ \epsilon$ *and* $\phi_1 = \epsilon \circ \delta$*.*
2. *We define* Γ_1 *and* Φ_1*, that act on* $\mathcal{G}^\times$*, by* $\Gamma_1 = \Delta \circ \mathcal{E}$ *and* $\Phi_1 = \mathcal{E} \circ \Delta$*.*
3. *We define the operators* $\gamma \circledvee \Gamma_1$ *and* $\phi \circledvee \Phi_1$ *by respectively* $\gamma \circledvee \Gamma_1(X) = (\gamma_1(X^\bullet), \Gamma_1(X^\times))$ *and* $\phi \circledvee \Phi_1(X) = (\phi_1(X^\bullet), \Phi_1(X^\times))$ *for any* $X \in \mathcal{G}$*.*

Figs. 2b and 2f present the result of $\gamma \circledvee \Gamma_1$ and $\phi \circledvee \Phi_1$ for respectively the subgraph of Fig. 2a and the one of Fig. 2e.

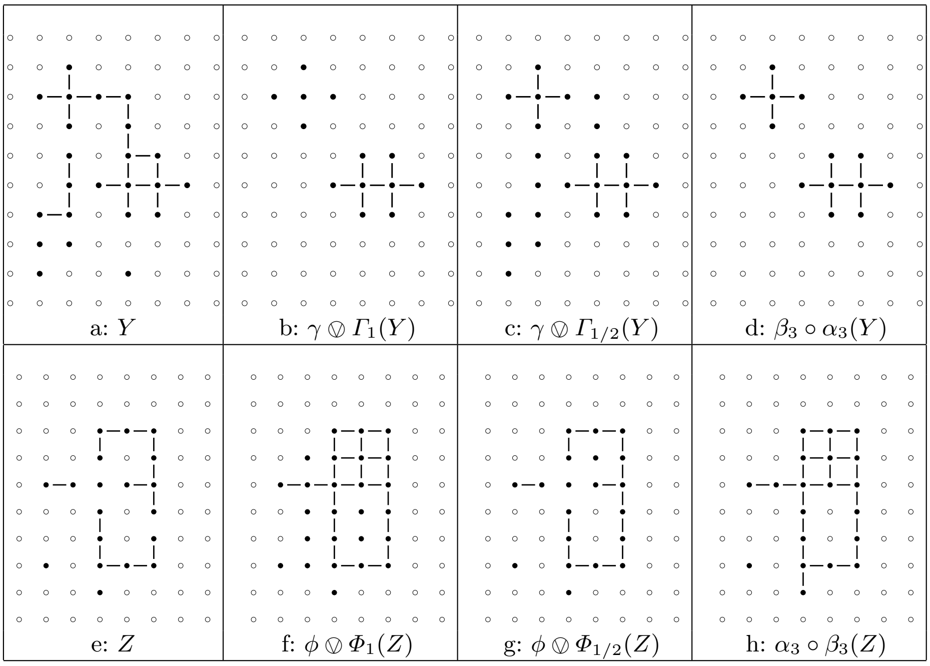

Fig. 2. Openings and closings ($\mathbb{G}$ is induced by the 4-adjacency relation).

In fact, by composing $\delta^\bullet$ with $\epsilon^\times$ and $\delta^\times$ with $\epsilon^\bullet$, we obtain smaller filters.

Definition 11 (half-opening, half-closing)

1. *We define* $\gamma_{1/2}$ *and* $\phi_{1/2}$*, that act on* $\mathcal{G}^\bullet$*, by* $\gamma_{1/2} = \delta^\bullet \circ \epsilon^\times$ *and* $\phi_{1/2} = \epsilon^\bullet \circ \delta^\times$*.*
2. *We define* $\Gamma_{1/2}$ *and* $\Phi_{1/2}$*, that act on* $\mathcal{G}^\times$ *by* $\Gamma_{1/2} = \delta^\times \circ \epsilon^\bullet$ *and* $\Phi_{1/2} = \epsilon^\times \circ \delta^\bullet$*.*
3. *We define the operators* $\gamma \circledvee \Gamma_{1/2}$ *and* $\phi \circledvee \Phi_{1/2}$ *by respectively* $\gamma \circledvee \Gamma_{1/2}(X) = (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times))$ *and* $\phi \circledvee \Phi_{1/2}(X) = (\phi_{1/2}(X^\bullet), \Phi_{1/2}(X^\times))$*, for any* $X \in \mathcal{G}$*.*

Thanks to Property 3, the operators defined above can be locally characterized. Let $X^\bullet \subseteq \mathbb{G}^\bullet$ and $Y^\times \subseteq \mathbb{G}^\times$, we have:

$$\begin{aligned}
\gamma_{1/2}(X^\bullet) &= \{x \in X^\bullet \mid \exists e_{x,y} \in \mathbb{G}^\times \text{ with } y \in X^\bullet\} \\
&= X^\bullet \setminus \{x \in X^\bullet \mid \forall e_{x,y} \in \mathbb{G}^\times, y \notin X^\bullet\} \\
\Gamma_{1/2}(Y^\times) &= \{u \in \mathbb{G}^\times \mid \exists x \in u \text{ with } \{e_{x,y} \in \mathbb{G}^\times\} \subseteq Y\} \\
&= Y^\times \setminus \{u \in Y^\times \mid \forall x \in u, \exists e_{x,y} \in \mathbb{G}^\times \text{ with } e_{x,y} \notin Y^\times\} \\
\phi_{1/2}(X^\bullet) &= \{x \in \mathbb{G}^\bullet \mid \text{ either } x \in X^\bullet \text{ or } \forall e_{x,y} \in \mathbb{G}^\times, y \in X^\bullet\} \\
&= X^\bullet \cup \{x \in \overline{X^\bullet} \mid \forall e_{x,y} \in \mathbb{G}^\times, y \in X^\bullet\} \\
\Phi_{1/2}(Y^\times) &= \{e_{x,y} \in \mathbb{G}^\times \mid \exists e_{x,z} \in Y^\times \text{ and } \exists e_{y,w} \in Y^\times\} \\
&= Y \cup \{e_{x,y} \in \overline{Y^\times} \mid x \in \delta^\bullet(Y^\times) \text{ and } y \in \delta^\bullet(Y^\times)\}.
\end{aligned}$$

Informally speaking, $\gamma_{1/2}$ removes from $Y^\bullet$ its isolated vertices whereas $\Gamma_{1/2}$ removes from $Y^\times$ the edges which do not contain a vertex completely covered by edges in $Y^\times$. It may be furthermore seen that $\gamma_{1/2}$ (resp. $\Gamma_{1/2}$) is the dual of $\phi_{1/2}$ (resp. $\Phi_{1/2}$). Thus, $\phi_{1/2}$ adds to $Y^\bullet$ the vertices of $\overline{X^\bullet}$ completely surrounded by elements of $Y^\bullet$ whereas $\Phi_{1/2}$ adds to $Y^\times$ the edges of $\overline{Y^\times}$ whose two extremities belong to at least one edge in $Y^\times$ (see for instance Fig. 2).

The family $\mathcal{G}$ is closed under the operators presented in Definition 10.3 since they are obtained by composition of operators also satisfying this property (Lemma 8). Furthermore, it can be deduced from the local characterization of the operators $\gamma_{1/2}$, $\Gamma_{1/2}$, $\phi_{1/2}$ and $\Phi_{1/2}$ that the family $\mathcal{G}$ is also closed under the operators of Definition 11.3. Hence, thanks to the properties of adjunctions recalled in the introduction of this section, the following theorem can be established.

Theorem 12 (graph-openings, graph-closings)

1. *The operators* $\gamma_{1/2}$ *and,* γ_1 *(resp.* $\Gamma_{1/2}$ *and* Γ_1*) are openings on* $\mathcal{G}^\bullet$ *(resp.* $\mathcal{G}^\times$*) and* $\phi_{1/2}$*, and* Φ_1 *(resp.* $\Phi_{1/2}$ *and* ϕ_1*) are closings on* $\mathcal{G}^\bullet$ *(resp.* $\mathcal{G}^\times$*).*
2. *The family* $\mathcal{G}$ *is closed under* $\gamma \circledvee \Gamma_{1/2}$*,* $\phi \circledvee \Phi_{1/2}$*,* $\gamma \circledvee \Gamma_1$*, and* $\phi \circledvee \Phi_1$*.*
3. *The operators* $\gamma \circledvee \Gamma_{1/2}$ *and* $\gamma \circledvee \Gamma_1$ *are openings on* $\mathcal{G}$ *and* $\phi \circledvee \Phi_{1/2}$ *and* $\phi \circledvee \Phi_1$ *are closings on* $\mathcal{G}$*.*

Composing the operators of the adjunctions (α_i, β_i), defined at the end of Section 3, also yields remarkable openings and closings. Indeed, it can be easily seen that: $\alpha_1 \circ \beta_1 = \alpha_1$, $\alpha_2 \circ \beta_2 = \alpha_2$, $\beta_1 \circ \alpha_1 = \beta_1$ and $\beta_2 \circ \alpha_2 = \beta_2$. Thus α_1

and α_2 are both a closing and an erosion and β_1 and β_2 are both a dilation and an opening. This means, in particular, that α_1 and α_2 are idempotent extensive erosions and that β_1 and β_2 are idempotent anti-extensive dilations. The opening and the closing resulting from the adjunction (α_3, β_3) are illustrated in Figs. 2d and 2h.

It is possible to associate with any lattice $\mathcal{L}$, the lattice of all increasing operators on $\mathcal{L}$. In this context, two filters φ_1 and φ_2 on the lattice $\mathcal{L}$ are said *ordered* if, for any $X \in \mathcal{L}$, $\varphi_1(X) \leq \varphi_2(X)$ or if, for any $X \in \mathcal{L}$, $\varphi_2(X) \leq \varphi_1(X)$. A usual way to build a hierarchy of filters (*i.e.* an ordered family of filters) from an adjunct pair (α, β) of erosion and dilation consists of building the dilations and erosions obtained by iterating several times α and β. In general, composing these iterated versions of α and β leads to hierarchies of filters when the number of iterations increases. In the remaining of the section, we follow this classical approach to build granulometries and alternate sequential filters in the lattice $\mathcal{G}$.

Let α be an operator acting on a lattice $\mathcal{L}$ and i be a nonnegative integer. The operator α^i is defined by the identity on $\mathcal{L}$ when $i = 0$ and by $\alpha \circ \alpha^{i-1}$ otherwise.

Definition 13 (granulometries, ASF) *Let $\lambda \in \mathbb{N}$.*

1. *We define $\gamma \circledcirc \Gamma_{\lambda/2}$ (resp. $\phi \circledcirc \Phi_{\lambda/2}$) by $\gamma \circledcirc \Gamma_{\lambda/2} = (\delta \circledcirc \Delta)^i \circ (\gamma \circledcirc \Gamma_{1/2})^j \circ (\epsilon \circledcirc \mathcal{E})^i$ (resp. $\phi \circledcirc \Phi_{\lambda/2} = (\epsilon \circledcirc \mathcal{E})^i \circ (\phi \circledcirc \Phi_{1/2})^j \circ (\delta \circledcirc \Delta)^i$), where i and j are respectively the quotient and the remainder in the integer division of λ by 2.*
2. *We define the operator $ASF_{\lambda/2}$ by the identity on graphs when $\lambda = 0$ and by $ASF_{\lambda/2} = \gamma \circledcirc \Gamma_{\lambda/2} \circ \phi \circledcirc \Phi_{\lambda/2} \circ ASF_{(\lambda-1)/2}$ otherwise.*

Note that it is possible to define a second family of operators similar to $ASF = \{ASF_{\lambda/2} \mid \lambda \in \mathbb{N}\}$ by replacing in Definition 13.2 the sequence of primitives $\gamma \circledcirc \Gamma_{\lambda/2} \circ \phi \circledcirc \Phi_{\lambda/2}$ by the sequence $\phi \circledcirc \Phi_{\lambda/2} \circ \gamma \circledcirc \Gamma_{\lambda/2}$. The following proposition (Property 14.2), which establishes that ASF is a family of alternate sequential filters, also holds true for this second family.

Property 14

1. *The families $\{\gamma \circledcirc \Gamma_{\lambda/2} \mid \lambda \in \mathbb{N}\}$ and $\{\phi \circledcirc \Phi_{\lambda/2} \mid \lambda \in \mathbb{N}\}$ are granulometries:*
   - *for any $\lambda \in \mathbb{N}$, $\gamma \circledcirc \Gamma_{\lambda/2}$ (resp. $\phi \circledcirc \Phi_{\lambda/2}$) is an opening (resp. a closing) on $\mathcal{G}$;*
   - *for any two elements $\lambda, \mu \in \mathbb{N}$ such that $\lambda \leq \mu$, we have $\gamma \circledcirc \Gamma_{\mu/2}(X) \sqsubseteq \gamma \circledcirc \Gamma_{\lambda/2}(X)$ and $\phi \circledcirc \Phi_{\lambda/2}(X) \sqsubseteq \phi \circledcirc \Phi_{\mu/2}(X)$ for any $X \in \mathcal{G}$.*
2. *The family $\{ASF_{\lambda/2} \mid \lambda \in \mathbb{N}\}$ is a family of alternate sequential filters:*
   - *for any two elements $\lambda, \mu \in \mathbb{N}$, $\lambda \geq \mu$ implies that $ASF_{\lambda/2} \circ ASF_{\mu/2} = ASF_{\lambda/2}$.*

Given a graph $X \in \mathcal{G}$, it must be noticed that the vertex set (resp. edge set) of $\gamma \circledcirc \Gamma_{\lambda/2}(X)$ and $\phi \circledcirc \Phi_{\lambda/2}(X)$ depends only on the vertex set (resp. edge set) of X. Thus, Definition 13 also induces granulometries and alternate sequential filters on $\mathcal{G}^\bullet$ and $\mathcal{G}^\times$. If we consider the case where λ is even, we deduce from the observation stated after Definition 5 that, when $\mathbb{G}^\bullet$ is a subset of the grid points $\mathbb{Z}^d$ and when $\mathbb{G}^\times$ is obtained from a symmetrical structuring element, then the vertex parts of $\gamma \circledcirc \Gamma_{\lambda/2}$ and $\phi \circledcirc \Phi_{\lambda/2}$ correspond to the usual opening and closing of size $\lambda/2$. Hence, we can see that, in the case of a set of points, the proposed framework completes the granulometries and filters which are classically used in applications by considering the odd values of λ.

In order to illustrate the proposed framework, let us analyze the effect of our filters on the binary image of Fig. 3b obtained by adding random impulse noise of different size and shape to the digital shape shown in Fig. 3a. Fig. 3c shows the results given by the "classical" ASF (using the structuring elements corresponding to the 4-adjacency relation) of size 4. Fig. 3d presents the results given by $ASF_{8/2}$ (on the graph induced by the 4-adjacency relation) which is the corresponding alternate filter in our framework. Clearly, $ASF_{8/2}$ removes more noise than the classical ASF. However, it requires more iterations since it considers the filters $\gamma \circledcirc \Gamma_{\lambda/2}$ and $\phi \circledcirc \Phi_{\lambda/2}$ for both odd and even values of λ whereas the classical ASF only considers the even values of λ. In order to compare the proposed ASF with filters using the same number of iterations, we produce two other filtered images. The first one is obtained by filtering the image with a classical ASF of size 8 (Fig. 3e). It can be seen that more noise are removed but also that less details are preserved (see in particular the head of the zebra). The second one (Fig. 3f) is obtained in three steps: 1) double the resolution of the noisy image; 2) apply to it a classical ASF of size 8; and 3) divide by two the resolution of the output image. It can be seen that this last procedure removes more noise than the classical ASF but does not perform as well as the ASF introduced in the present paper. This qualitative assessment is confirmed by a quantitative study that will be published in a future extended version of this paper [10]. Fig. 4 provides a similar illustration for the case of a 3-dimensional synthetic binary object.

5 Conclusion

This paper investigates the lattice of all subgraphs of a graph and provides it with morphological operators. In particular, we propose new filters which input and output graphs. We show the interest of restricting these filters to sets of vertices. Indeed, they allow us to complete some classical morphological filters used in image analysis applications.

In future work, the proposed approach will be extended to (node and edge) weighted graphs considered as stacks of graphs. We will also study morphology in simplicial (and cubical) complexes (see [2] for image operators defined in cubical complexes and [16] for examples of morphological operators in 2D simplicial complexes). These topological structures extend graphs to higher dimensions in the sense that a graph is a 1-D structure made of points and edges considered as 0D and 1D elements. The proposed approach extends to general complexes by considering additional generators. In 2D, for instance, a third generator for the elementary triangles (or squares) is required.

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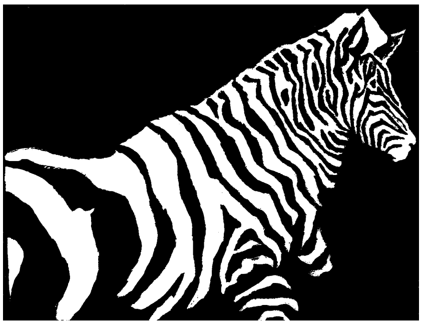
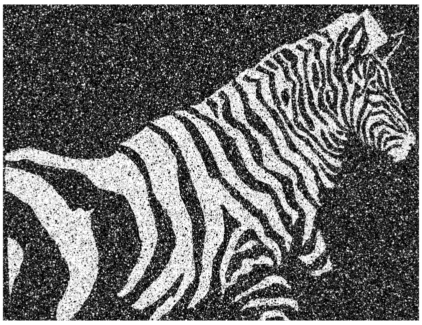


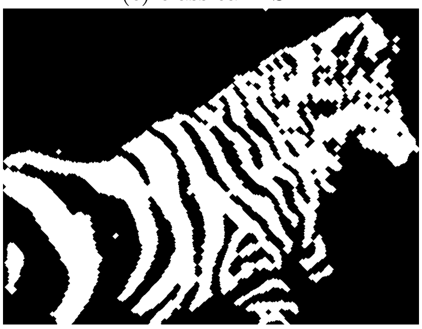
(a) original binary image



(b) noisy binary image

(c) classical ASF

(d) graph ASF



(e) classical ASF of double size

(f) classical ASF (double resolution)

Fig. 3. ASF illustration [see text].

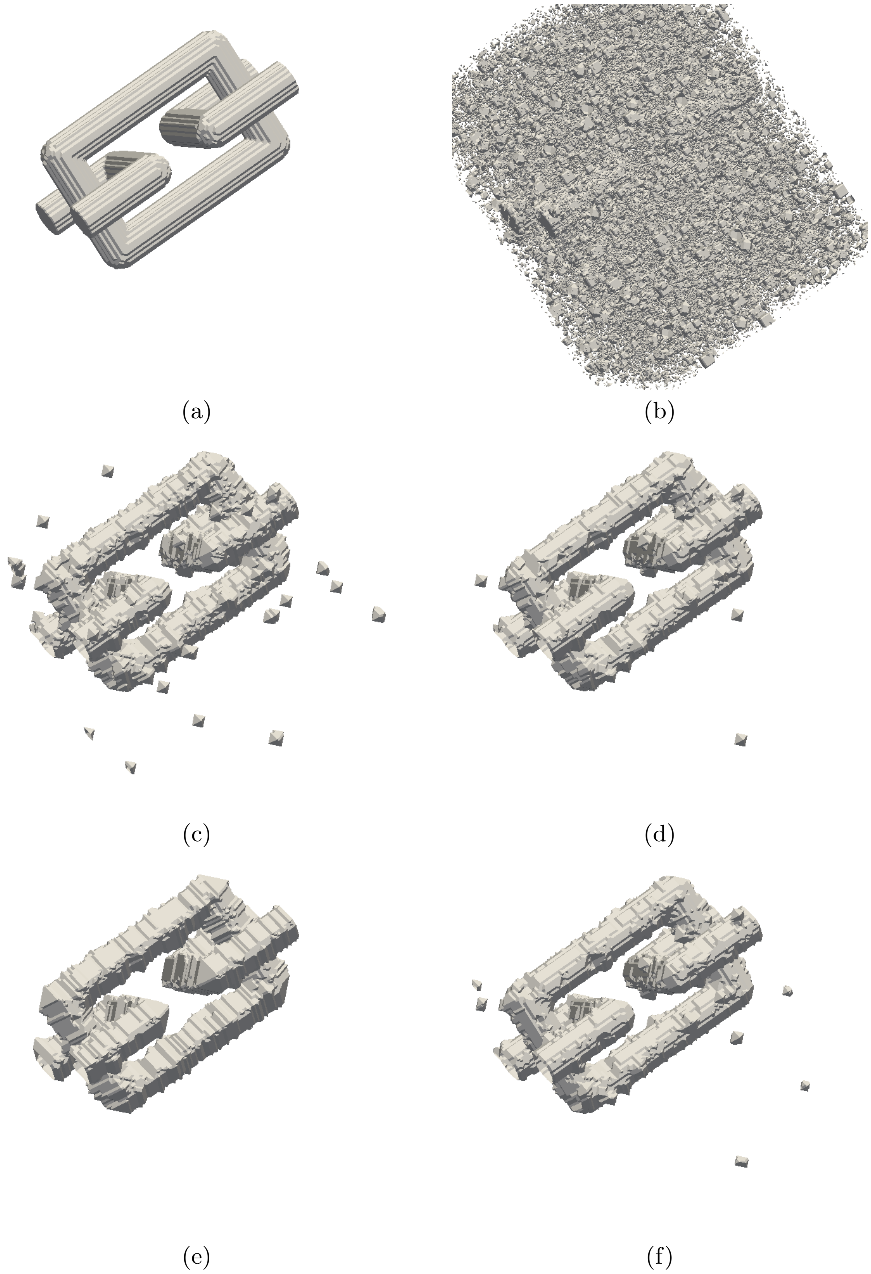


Fig. 4. Same as Fig. 3 but in 3D. Rendering of: (a) original binary image, (b) noisy binary image, (c) result of classical ASF, (d) result of graph ASF, (e) result of classical ASF of double size and (f) result of classical ASF (double resolution) [see text].