Simple points, critical kernels, and thinning algorithms

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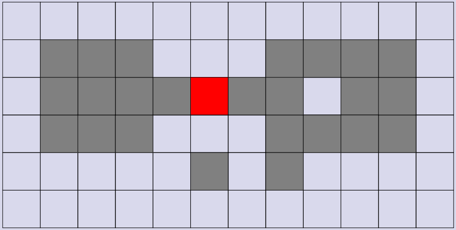
Plan of the presentation

- Combinatorial homotopy (informal)
- Simple points
- Critical Kernels
- Thinning algorithms

Homotopic thinning

Simple point

Intuitively, a point is simple if it can be removed without changing topology



non simple

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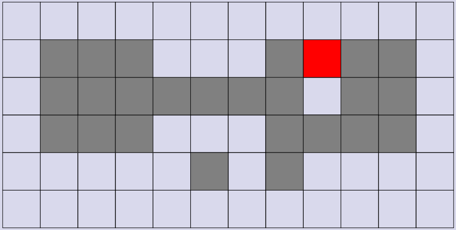

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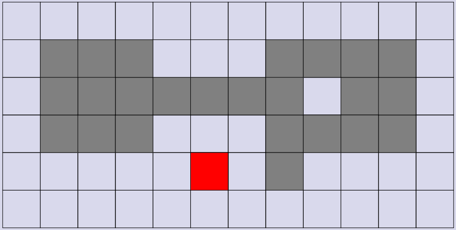

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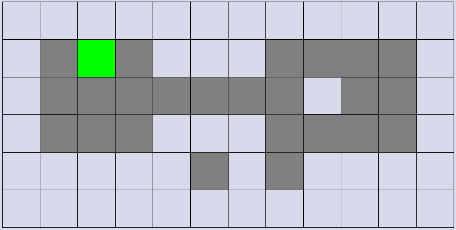

simple

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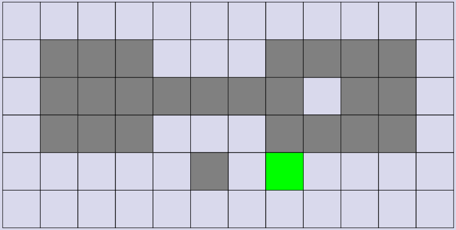


simple

Combinatorial homotopy (by simple points)

We say that an object Y is a retraction of an object X if Y may be obtained from X by a sequence of “simple point deletion”.

We say that an object Y is homotopic to an object X if Y may be obtained from X by a sequence of “simple point deletion‘” or “simple point addition”.

Simple point: the 2D case

Let X be a rectangle and let Y be a retraction of X. If Y has no simple point, then Y is an object which consists in a single point.

Rosenfeld (1970), C. Ronse (1986)

Confluence properties

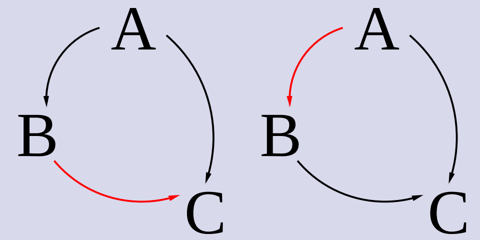
Let A, B, C be any three objects such that $C \subset B \subset A$ and let OP be an operator.

Downstream confluence property:

If $A \xrightarrow{OP} B$ and $A \xrightarrow{OP} C$, then $B \xrightarrow{OP} C$.

Upstream confluence property:

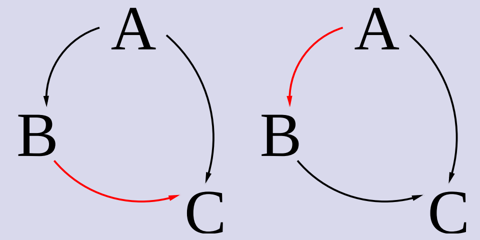
If $A \xrightarrow{OP} C$ and $B \xrightarrow{OP} C$, then $A \xrightarrow{OP} C$.



Topological watersheds, G. Bertrand (2005)

Simple points: confluence properties in 2D

Let A, B, C be any three objects such that $C \subset B \subset A$.
The two following confluence properties hold:



N. Passat, M. Couprie, L. Mazo, G. Bertrand (2010)

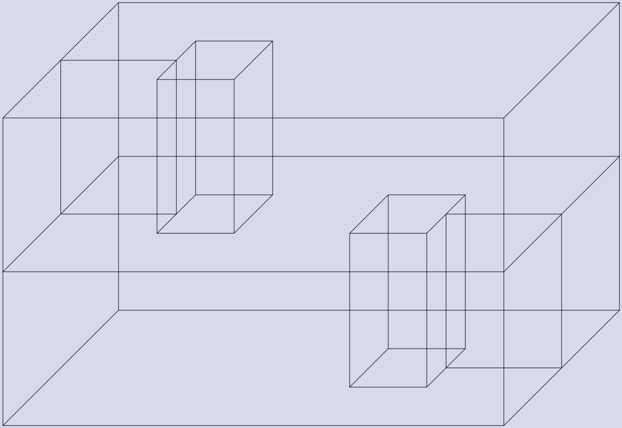
Simple points: confluence properties

The 3D case ?

Simple points: confluence properties

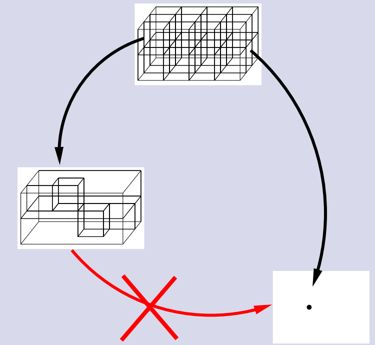
These confluence properties are not true in 3D !

Bing's house



Bing's house is a counter-example for 3D confluence

Bing's house has no simple point.



R.H. Bing (1964), E.C. Zeeman (1964)

Bing's house is a counter-example for 3D confluence

Such configurations may appear in real images !

Percentage p of "pathological objects" obtained by randomly generating 10 000 skeletons from a $N \times N \times N$ cube.

N	10	20	30	40
p	0,0001	0,0249	0,1739	0,5061

N. Passat, M. Couprie, G. Bertrand (2008)

Combinatorial homotopy

A notion which is not easy to handle !

- NP-complete problems
- Undecidable problems
- Poincaré “conjecture”

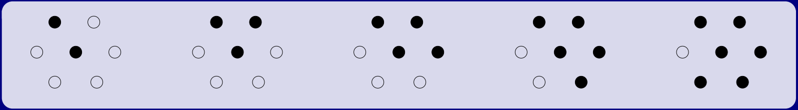
A. A. Markov (1958), R. Malgouyres and A. R. Francés (2008)

Simple points

Characterization of simple points

The 2D case:

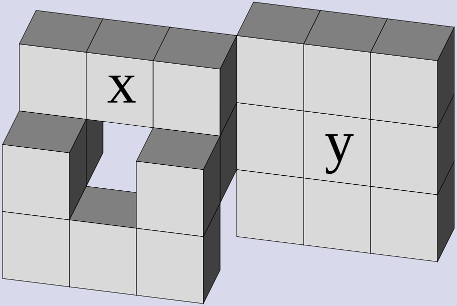
- The notion of connectedness (for both the object and the background) suffices to characterize simple pixels.
- Simple pixels may be characterized by a set of masks.



Characterization of simple points

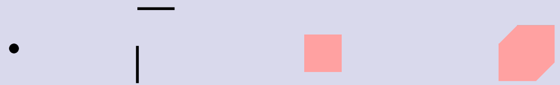
The 3D case:

Things are more difficult !



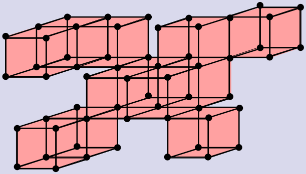
Cubical complexes

A complex is a set of faces which are "glued together".



0-face 1-faces 2-face (pixel) 3-face (voxel)

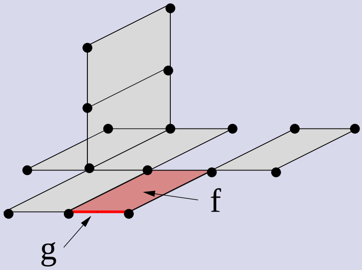
A complex:



H. Poincaré (1895)

Elementary collapse

- Let f and g be two distinct faces such that f is the only face of X which contains g.
- The complex $X \setminus \{f, g\}$ is an elementary collapse of X.



J.H.C. Whitehead (1939)

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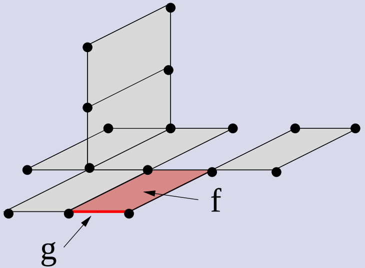

Observe that f is necessarily a facet, *i.e.*, f is maximal for inclusion.

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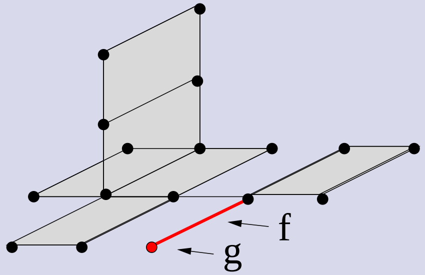

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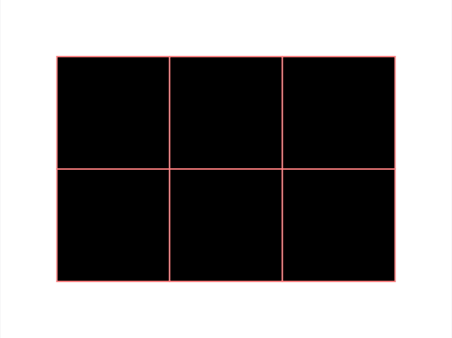
Collapse preserves topology



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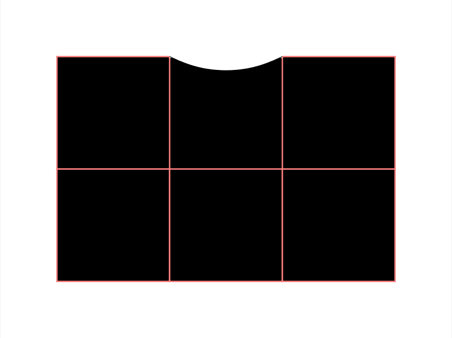

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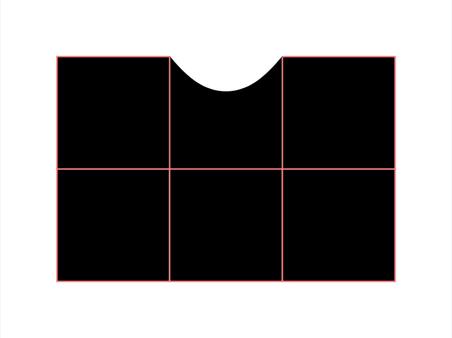

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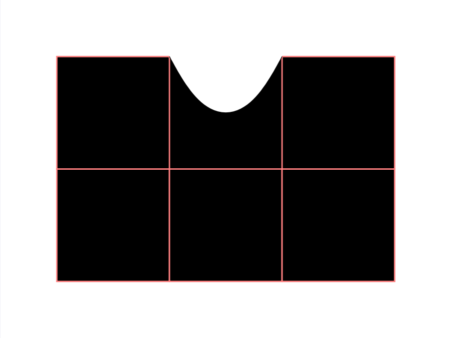

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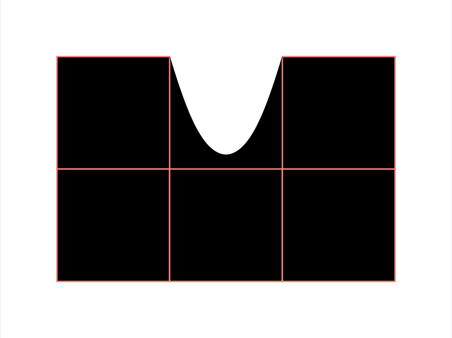

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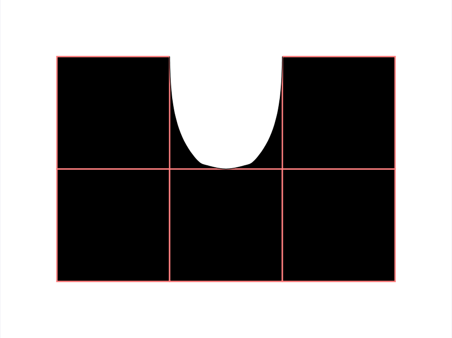

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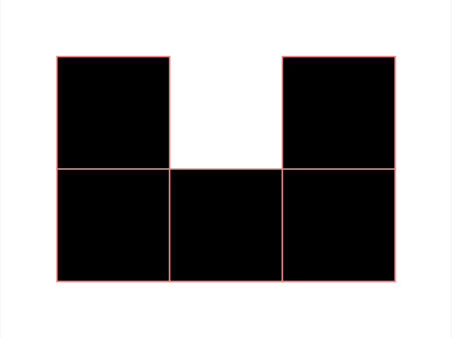
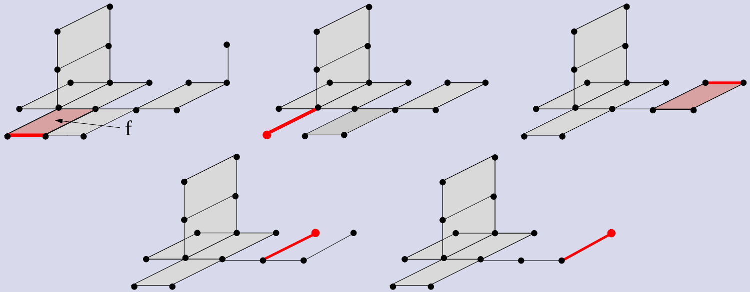


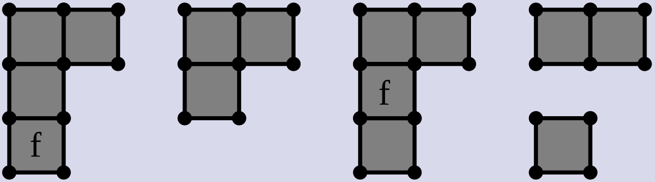
Collapse sequence

- Let X, Y be two complexes. We say that X collapses onto Y if there exists a collapse sequence from X to Y.



Detachment

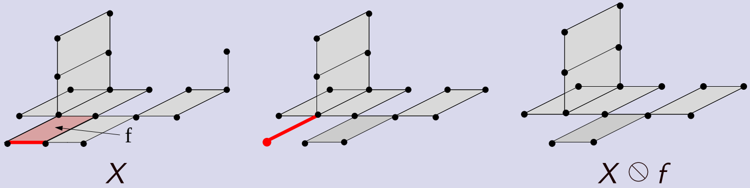
The detachment operation (denoted by $\oslash$) "removes" a facet from a complex, yielding a new complex.



X $\quad X \oslash f$ $\quad X$ $\quad X \oslash f$

Definition of simple facets

Let f be a facet, we say that f is simple if X collapses onto $X \oslash f$.



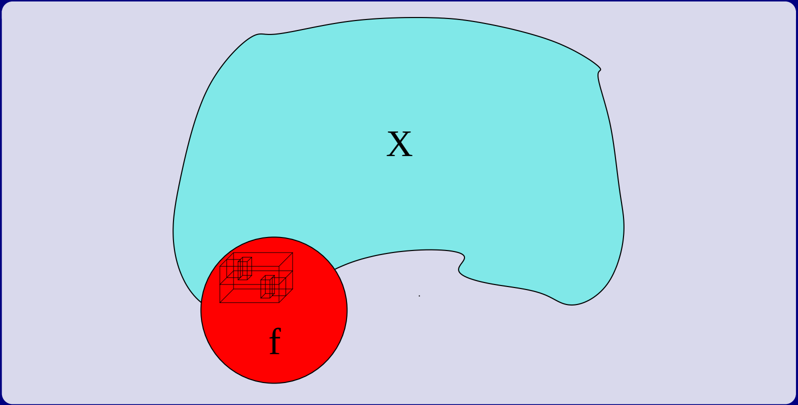
The facet f is simple.

G. Bertrand (2007)

Definition of simple facets

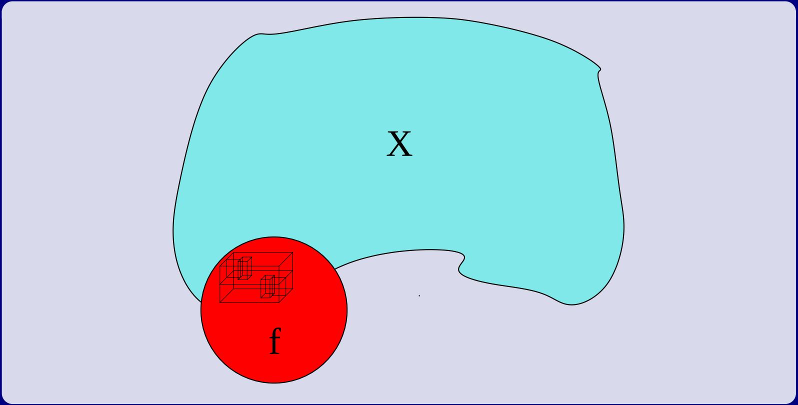
This new definition generalizes all previous definitions.
It works for objects of arbitrary dimensions.

Higher dimensions



$f \cap X$ is a Bing's house and f is simple for $X \cup f$

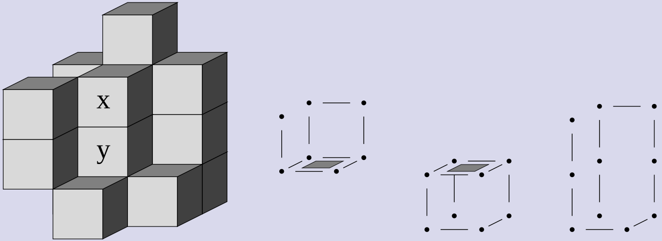
Higher dimensions



M. Couprie and G. Bertrand (2009): this kind of configuration may happen in 5D (but not in 4D) !

Extension of simple points: simple pairs

The voxels x and y are both not simple. Nevertheless we can remove x and y without changing the topology of the object.

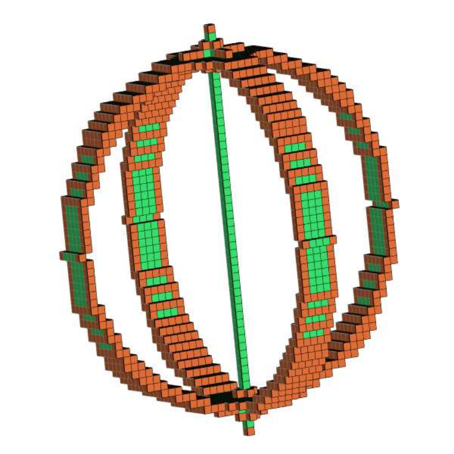


N. Passat, M. Couprie and G. Bertrand (2008)

Critical kernels

A framework for the study of parallel thinning

Parallel removal of simple points may alter topology.



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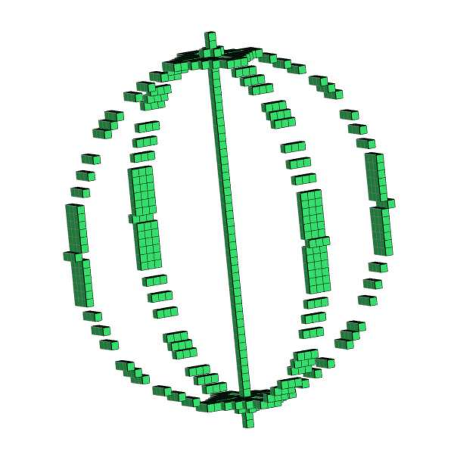
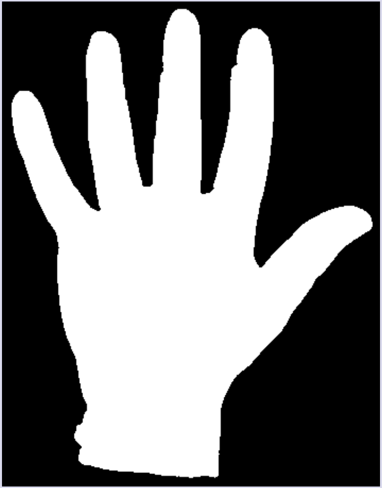


Milestones

- 1966: D. Rutovitz – first parallel thinning algorithm
- 1970: A. Rosenfeld – digital topology
- 1988: C. Ronse – minimal non-simple sets
- 1995: G. Bertrand – P-simple points
- 2005: G. Bertrand – Critical kernels

Motivation

- To be able to thin objects in a regular and symmetric way.
- To be able to have well-defined skeletons, *i.e*, to have skeletons which are unique and which do not depend on the order of the simple points selection.
- Of course, parallelism may also be used for faster computations.



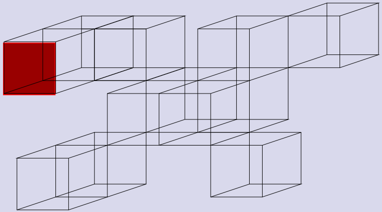
Critical kernels: key notions

This framework is based on only three notions:

- Essential face
- Core of a face
- Regular/critical face

Essential face

We say that f is an essential face if f is precisely the intersection of all facets of X which contain f.
Note: Any facet is essential.



The 2-face is not essential

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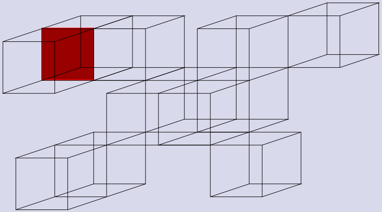

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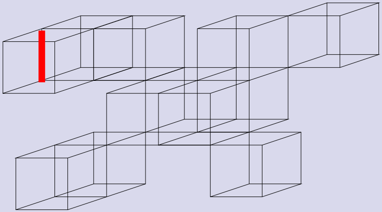

The 1-face is not essential

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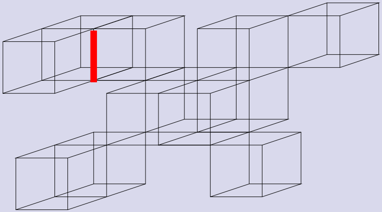

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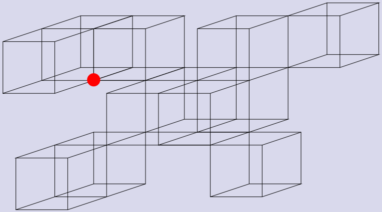

The 0-face is not essential

Essential face

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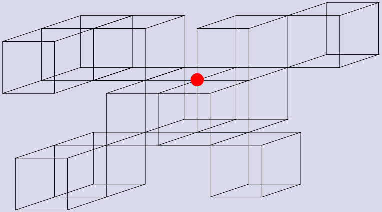
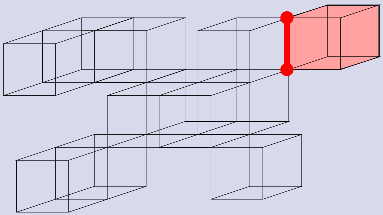


The 0-face is essential

Core

- The core of f is the complex, denoted by $Core(f, X)$, composed by all the essential faces which are strictly included in f, and all the faces included in these faces.



A 3-face and its core

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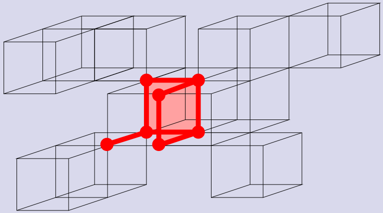

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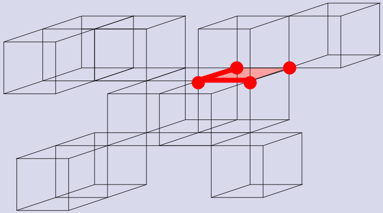
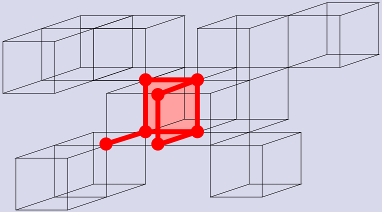


A 2-face and its core

Regular/critical face

- We say that f is regular if f is essential and if $\hat{f}$ collapses onto $Core(f, X)$.
- We say that f is critical if f is essential and not regular.



The 3-face is regular

Regular/critical face

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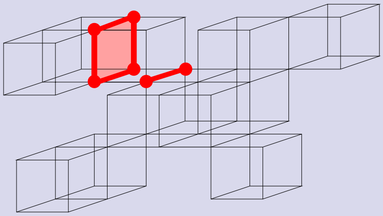

The 3-face is critical

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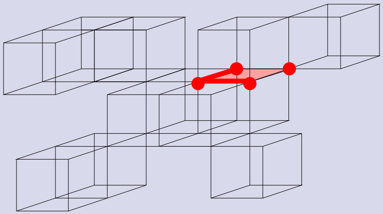
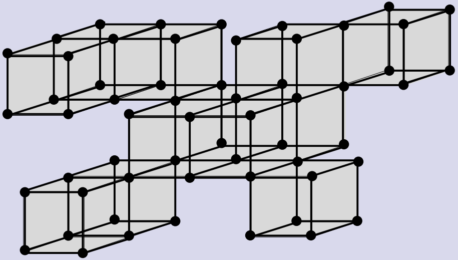


The 2-face is critical

Critical kernel

- We set $Critic(X) = \cup\{\hat{f} \mid f \text{ is critical }\}$, $Critic(X)$ is the critical kernel of X.



A complex X

Critical kernel

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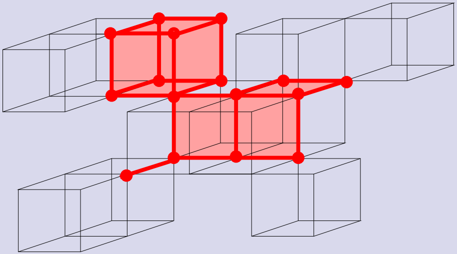
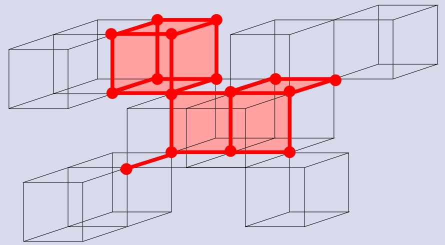


The critical kernel of X

Main theorem

In any dimension, X collapses onto the critical kernel of X.

Furthermore, if Y is any set of facets of X such that Y contains the critical kernel of X, then X collapses onto Y.



G. Bertrand (2007)

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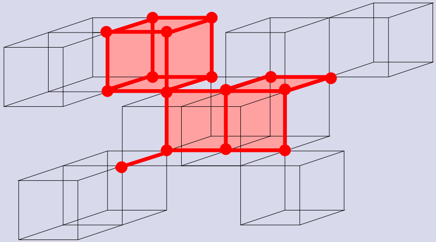
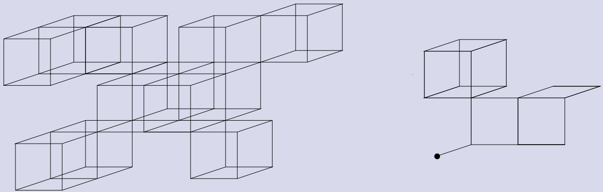


This theorem leads to a wide class of topologically correct n-D parallel thinning algorithms, based on the different possible choices of the set Y.

Crucial kernels: motivation

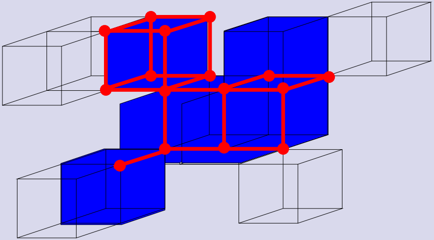
The critical kernel of a set of voxels is not always a set of voxels



In the following, we assume that X is a set of voxels (*i.e.*, a complex in which each principal face is a 3-face).

Crucial kernel

The crucial kernel of a complex X is the set of all facets of X which contain a maximal face of the critical kernel of X. Thus X collapses onto its crucial kernel.



Thinning algorithms

Constrained $\mathcal{K}$-skeleton

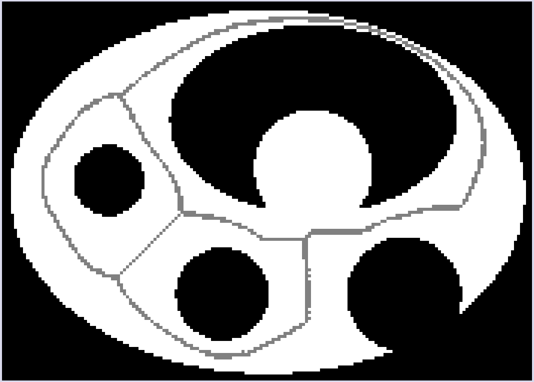
Definition

- Let S be a set of n-xels and let $K \subseteq S$.
- We denote by $Cruc(S, K)$ the set composed of all n-xels which are in the crucial kernel of S or which are in K.
- Let $\langle S_0, S_1, \ldots, S_k \rangle$ be the unique sequence such that $S_0 = S$, $S_i = Cruc(S_{i-1}, K)$, $i = 1, \ldots, k$ and $S_k = Cruc(S_k, K)$.
- The set S_k is the $\mathcal{K}$-skeleton of S constrained by K.

G. Bertrand and M. Couprie (2008)

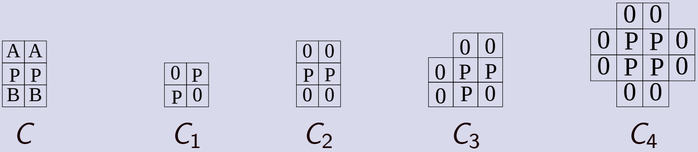
Minimal $\mathcal{K}$-skeleton

We set $K = \emptyset$



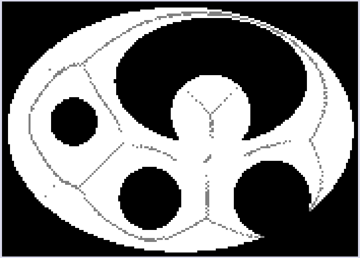
As far as we know, this is the first fully parallel algorithm for minimal skeletons

Local conditions (2D) for crucial pixels

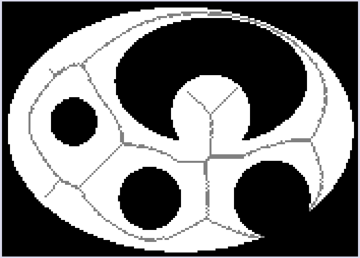


$\mathcal{K}$-skeleton constrained by the medial axis

We set $K = \text{MedialAxis}(X)$

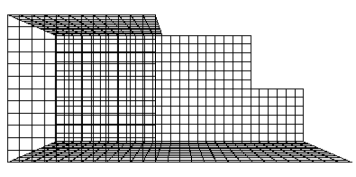


Medial axis

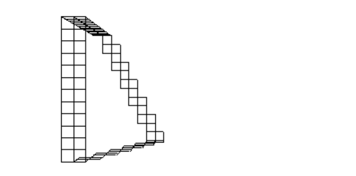


Constrained skeleton

Minimal 2D $\mathcal{K}$-skeleton in the 3D grid



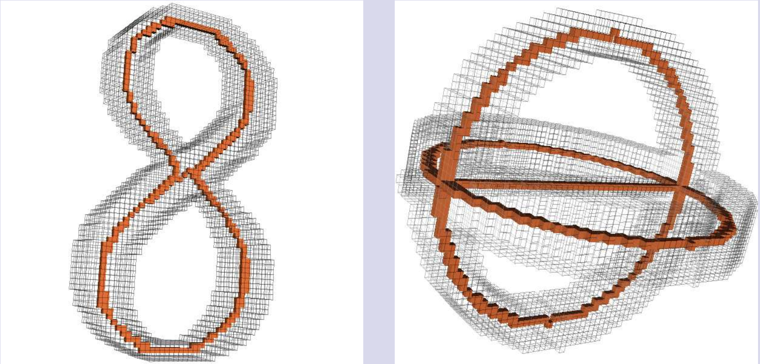
Object S



Minimal $\mathcal{K}$-skeleton of S

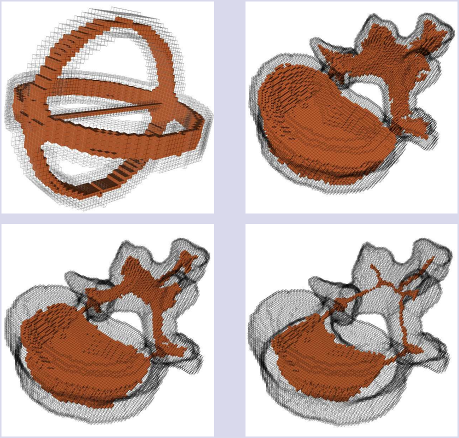
G. Bertrand and M. Couprie (2008): 10 new 2D parallel thinning algorithms

3D Skeletons: minimal skeleton



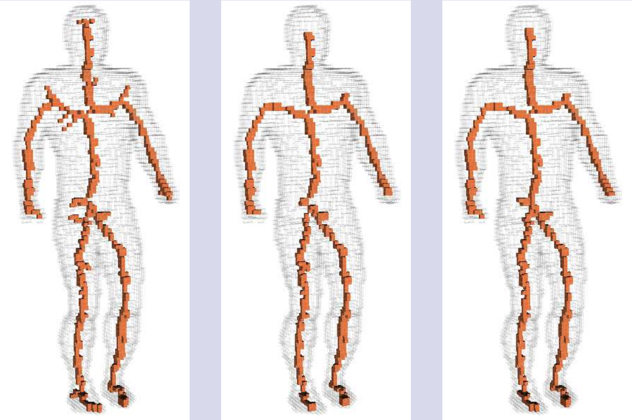
G. Bertrand and M. Couprie (2006): New 3D parallel thinning algorithms

3D Skeletons: surface skeletons



G. Bertrand and M. Couprie (2006): New 3D parallel thinning algorithms

3D Skeletons: curvilinear skeletons



G. Bertrand and M. Couprie (2006): New 3D parallel thinning algorithms

Conclusion

- A new definition of simple points
- New sound 2D and 3D thinning algorithms
- Order independent skeletons
- A generic thinning scheme
- Analysis of existing thinning algorithms
- A generalization of minimal non-simple sets and P-simple points

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