Short note on natural cubic splines and discrete curves

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Abstract

This note gives information about how to commpute the coefficients of a natural cubic spline that approximates a given discrete curve.

1 Definition

Let $x_1 < \ldots < x_n$ be n real numbers. Let $a, b \in \mathbb{R}$ be such that $a < x_1$ and $b > x_n$. Let f be a function from $[a, b]$ into $\mathbb{R}$. For all $i \in \{1, \ldots, n\}$, we write $f(x_i) = y_i$ (see Fig. 1).

The function f is a natural cubic spline with nodes $(x_1, y_1), \ldots, (x_n, y_n)$ if it satisfies the following conditions:

- The function f is of class C^2.
- The restrictions $f/[a, x_1]$ and $f/[x_n, b]$ coincide with polynoms of degree less than or equal to 1.
- For all $i \in \{1, \ldots, n-1\}$, the restriction $f/[x_i, x_{i+1}]$ coincides with a polynom of degree less than or equal to 3.

2 Computation of a spline from control points

Our input is a list of control points: $(x_1, y_1), \ldots, (x_n, y_n)$. Our goal is to compute the coefficients of the polynoms that constitute a spline f with nodes $(x_1, y_1), \ldots, (x_n, y_n)$.

For any $i \in \{1, \ldots, n\}$ and any family $\{a_i\}_{i \in \mathbb{N}}$, we set:

$$z_i = f''(x_i) \quad \text{and} \quad \Delta a_i = a_{i+1} - a_i.$$

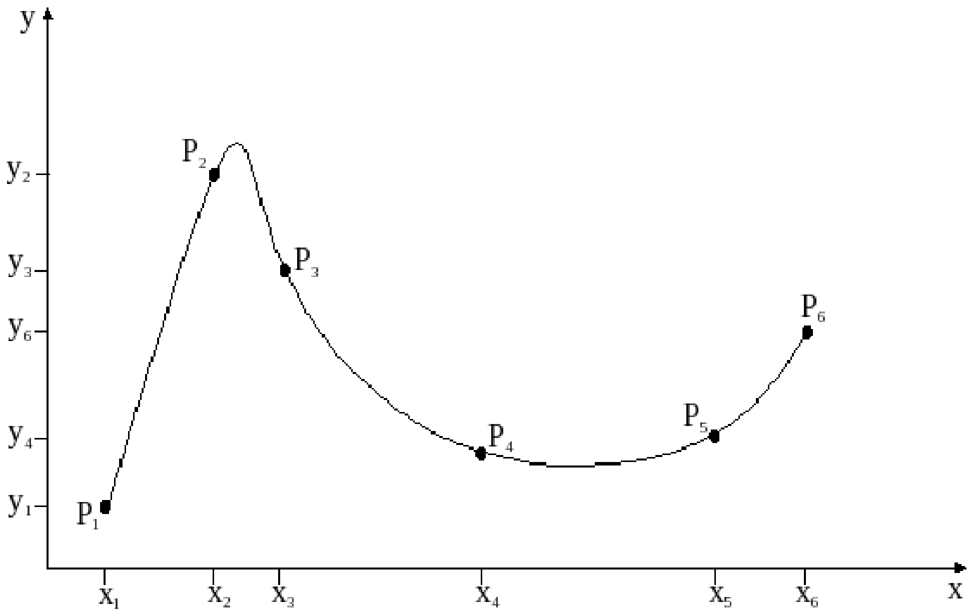

Fig. 1. A spline with 6 control points.

For all $j \in \{1, \dots, n-2\}$, for all $x \in [x_j, x_{j+1}]$,

$$f''(x) = \frac{x_{j+1} - x}{\Delta x_j} z_j + \frac{x - x_j}{\Delta x_j} z_{j+1}.$$

Let $(A_j, B_j) \in \mathbb{R}^2$ such that, for all $x \in [x_j, x_{j+1}]$,

$$f(x) = A_j + B_j(x - x_j) + \frac{(x_{j+1} - x)^3}{6\Delta x_j} z_j + \frac{(x - x_j)^3}{6\Delta x_j} z_{j+1}.$$

We have $f(x_j) = y_j$ and $f(x_{j+1}) = y_{j+1}$, thus:

$$A_j = y_j - \frac{z_j(\Delta x_j)^2}{6} \quad \text{and} \quad B_j = \frac{\Delta y_j}{\Delta x_j} - \frac{\Delta x_j \Delta z_j}{6}.$$

The continuity of f'' in x_1 and x_n implies:

$$z_1 = z_n = 0.$$

The continuity of f' in x_j implies that, for all $j \in \{2, \ldots, n-1\}$,

$$\frac{\Delta x_{j-1}}{6} z_{j-1} + (\frac{\Delta x_j}{3} + \frac{\Delta x_{j-1}}{3}) z_j + \frac{\Delta x_j}{6} z_{j+1} = \frac{\Delta y_j}{\Delta x_j} - \frac{\Delta y_{j-1}}{\Delta x_{j-1}}.$$

We obtain a system of linear equations that is represented by a tridiagonal matrix and can be easily solved.

3 Natural parametrization and discretization of a spline

The natural parametrization of f, taking as origin the first control point x_1, is given by:

$$\int_{x_1}^{x} \sqrt{1 + (f'(t))^2} dt.$$

We want to find a subdivision of the spline f by a new set $(x'_1, \ldots, x'_m)$ of m control points, such that all portions have equal lengths (regular discretization).

We set:

$$L = \int_{x_1}^{x_n} \sqrt{1 + (f'(t))^2} dt,$$

that is, the total length of the spline. Then, we have, for all $k \in \{1, \ldots, m\}$:

$$\int_{x_1}^{x'_k} \sqrt{1 + (f'(t))^2} dt = \frac{kL}{m}.$$

For all $k \in \{1, \ldots, m\}$, there exists a unique p in $\{1, \ldots, n-1\}$ such that

$$\int_{x_1}^{x_p} \sqrt{1 + (f'(t))^2} dt \leq \frac{kL}{m} < \int_{x_1}^{x_{p+1}} \sqrt{1 + (f'(t))^2} dt.$$

We find x'_k as the unique solution in $[x_p, x_{p+1}]$ of the equation:

$$\int_{x_p}^{x'_k} \sqrt{1 + (f'(t))^2} dt = \frac{kL}{m} - \int_{x_1}^{x_p} \sqrt{1 + (f'(t))^2} dt.$$

We can compute each x'_k by dichotomy.

For a parametric curve in $\mathbb{R}^d$, represented by d functions $f_1, \ldots, f_d$, the expression of the natural parametrization becomes:

$$\int_{x_1}^{x} \sqrt{(f'_1(t))^2 + \ldots + (f'_d(t))^2} dt.$$

Corresponding changes must be done in the computation of a subdivision described above.

4 Curvature

The curvature at point x is given by:

$$\frac{f''(x)}{(1 + (f'(x))^2)^{3/2}}$$

for the case of function f from $\mathbb{R}$ to $\mathbb{R}$.

It is expressed by:

$$\frac{f'_1(x)f''_2(x) - f'_2(x)f''_1(x)}{((f'_1(x))^2 + (f'_2(x))^2)^{3/2}}$$

for a parametric curve in $\mathbb{R}^2$, represented by functions f_1, f_2, and by

$$\frac{\sqrt{K_{12} + K_{13} + K_{23}}}{((f'_1(x))^2 + (f'_2(x))^2 + (f'_3(x))^2)^{3/2}},$$

where $K_{ij} = (f'_i(x)f''_i(x) - f'_j(x)f''_i(x))^2$, for a parametric curve in $\mathbb{R}^3$, represented by functions f_1, f_2, f_3.

5 Finding a spline that approximates a discrete curve

Let $(P_1, \ldots, P_m)$ be a sequence of points of $\mathbb{Z}^n$ (pixels or voxels) forming a discrete curve. Let T be a positive real number (tolerance).

Here, we consider parametric splines. A parametric spline in 2D is composed by two natural cubic splines f_1, f_2, the points of the parametric spline are the couples $(f_1(t), f_2(t))$ for convenient values of parameter t. The 3D case is similar with three functions f_1, f_2, f_3. Thus, computing a parametric spline in 2D (resp. 3D) amounts to compute two (resp. three) natural cubic splines, as described in Sec. 2.

Our goal is to find a subset of the points of the discrete curve such that the parametric spline interpolating these points lies "near" the discrete curve, with respect to the given tolerance T.

Our algorithm is the following (see Figs. 2 and 3 for illustrations).

Initialization:

Set a list L of control points: $L = (A_0, B, A_1)$ where $A_0 = P_1$, $A_1 = P_m$, and B is the median point of the discrete curve.

Loop:

Compute the parametric spline from the control points in L.

Between any two successive points in L, compute the maximal distance between the spline segment and the discrete curve segment. If this distance is greater than T, insert in L a new control point that is the median point of this discrete curve segment.

Loop until no new point is added to L during an iteration.

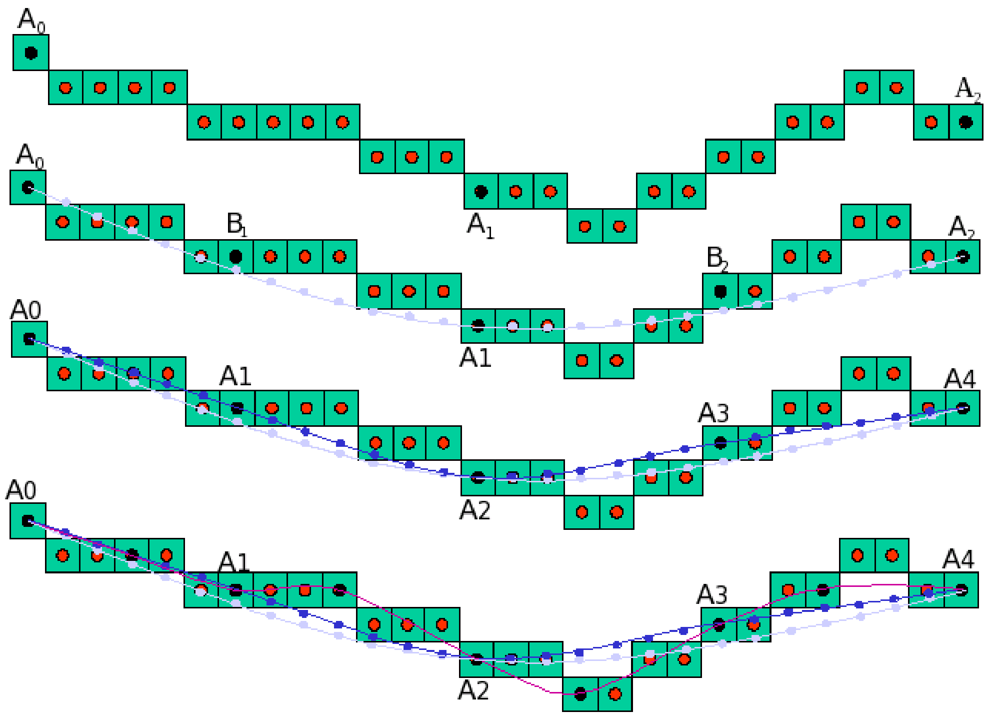

Fig. 2. Illustration (1) of the algorithm.

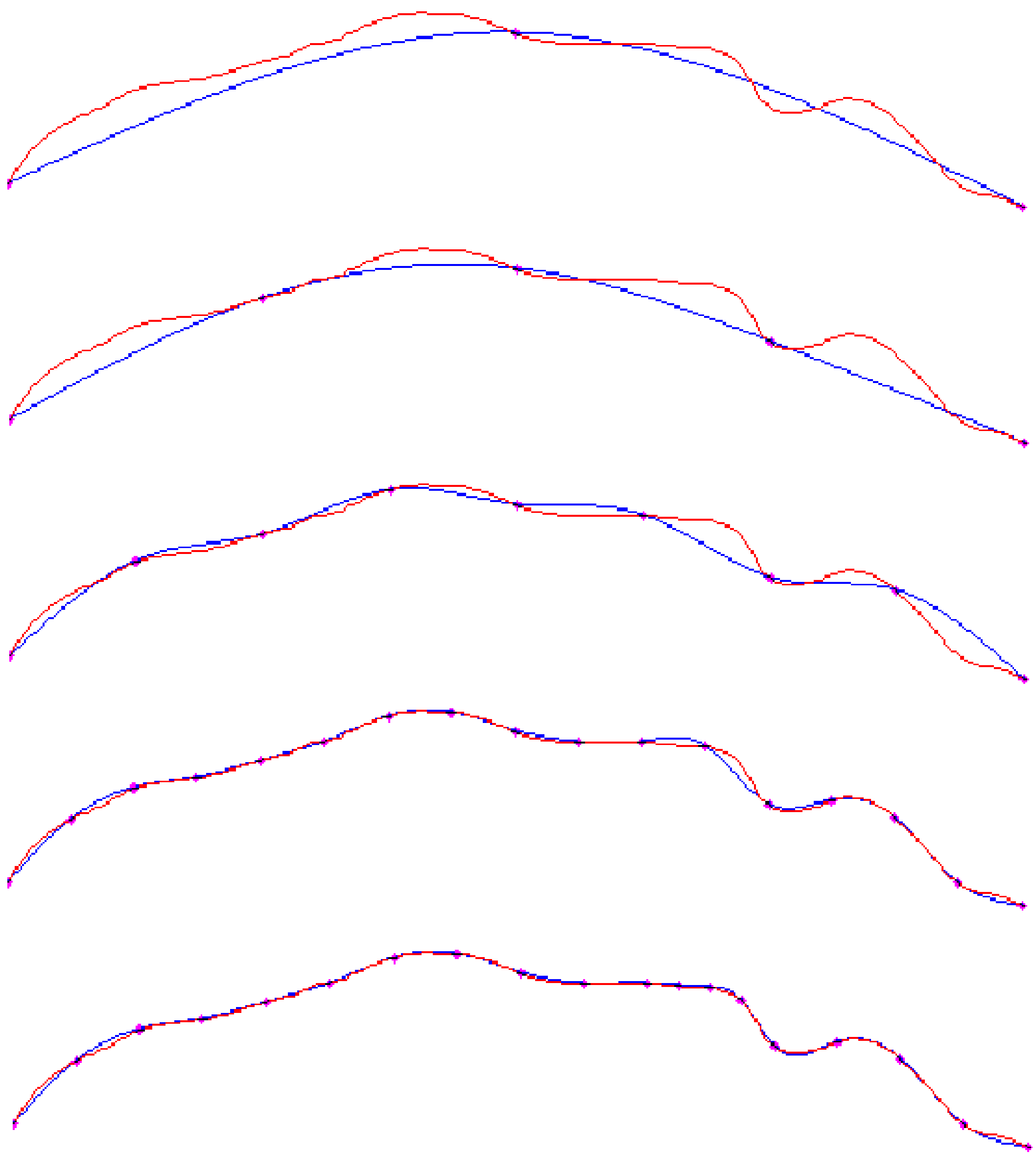

Fig. 3. Illustration (2) of the algorithm.