Minimal simple pairs in the cubic grid

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Abstract. Preserving topological properties of objects during thinning procedures is an important issue in the field of image analysis. This paper constitutes an introduction to the study of non-trivial simple sets in the framework of cubical 3-D complexes. A simple set has the property that the homotopy type of the object in which it lies is not changed when the set is removed. The main contribution of this paper is a characterisation of the non-trivial simple sets composed of exactly two voxels, such sets being called minimal simple pairs.

Key words: Cubical complexes, topology preservation, collapse, thinning, 3-D space.

1 Introduction

Topological properties are fundamental in many applications of image analysis. Topology-preserving operators, like homotopic skeletonisation, are used to transform an object while leaving unchanged its topological characteristics. In discrete grids ($\mathbb{Z}^2$ or $\mathbb{Z}^3$), such a transformation can be defined and efficiently implemented thanks to the notion of simple point [16]: intuitively, a point of an object is called simple if it can be deleted from this object without altering its topology.

A typical topology-preserving transformation based on simple points deletion, that we call *guided homotopic thinning* [9,8], may be described as follows. The input data consists of a set X of points in the grid (called object), and a subset K of X (called constraint set). Let $X_0 = X$. At each iteration i, choose a simple point x_i in X_i but not in K according to some criterion (*e.g.*, a priority function) and set $X_{i+1} = X_i \setminus \{x_i\}$. Continue until reaching a step n such that no simple point for X_n remains in $X_n \setminus K$. We call the result of this process a *homotopic skeleton of X constrained by K*. Notice that, since several points may have the same priority, there may exist several homotopic skeletons for a given pair X, K.

The most common example of priority function for the choice of x_i is a distance map which associates, to each point of X, its distance from the boundary of X. In this case, the points which are closest to the boundary are chosen first, resulting in a skeleton which is "centered" in the original object. In some particular applications, the priority function may be obtained through a greyscale

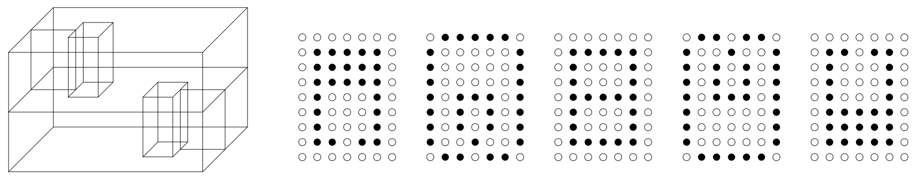


Fig. 1. Left: The Bing's house with two rooms. Right: A discrete version of the Bing's house, decomposed into its five planar slices for visualisation. The 26-adjacency relation is used for object points.

image, for example when the goal is to segment objects in this image while respecting topological constraints (see *e.g.* [10,22]). In the latter case, the order in which points are considered does not rely on geometrical properties, and may be affected by noise.

In such a transformation, the result is expected to fulfil a property of minimality, as suggested by the term "skeleton". This is indeed the case for the procedure described above, since the result X_n is minimal in the sense that it contains no simple point outside of K. However, we could formulate a stronger minimality requirement, which seems natural for this kind of transformation: informally, the result X_n should not strictly include any set Y which is "topologically equivalent" to X, and which contains K. We say that a homotopic skeleton of X constrained by K is *globally minimal* if it fulfils this condition.

Now, a fundamental question arises: is any homotopic skeleton globally minimal? Let us illustrate this problem in dimensions 2 and 3. In $\mathbb{Z}^2$, consider a full rectangle X of any size, and the constraint set $K = \emptyset$. Obviously, this object X is topologically equivalent to a single point, thus only homotopic skeletons which are singletons are globally minimal. A. Rosenfeld proved in [21] that any homotopic skeleton of X is indeed reduced to a single point.

But quite surprisingly, in dimension 3, this property does not hold: if X is *e.g.* a full $10 \times 10 \times 10$ cube, we may find a homotopic skeleton of X (with empty constraint set) which is not reduced to a single point. This fact constitutes one of the main difficulties when dealing with certain topological properties, such as the Poincaré conjecture. A classical counter-example is the Bing's house with two rooms [6], illustrated in Fig. 1 (left). One can enter the lower room of the house by the chimney passing through the upper room, and vice-versa. A discrete version B of the Bing's house is displayed in Fig. 1 (right). It can be seen that the Bing's house can be carved from a full cube by iterative removal of simple points. It can also be seen that B contains no simple point: deleting any point from B would create a "tunnel".

It could be argued that objects like Bing's houses are unlikely to appear while processing real (noisy) images, because of their complex shape and their size. However, we found that there exists a large class of objects presenting similar properties, some of them being quite small (less than 50 voxels). Let us call a *lump relative to* K any object X which has no simple point outside of K, and which strictly includes a subset Y including K and topologically equivalent to X (*i.e.*, a homotopic skeleton which is not globally minimal). This notion of lump is formalised and discussed in Appendix A. One of the authors detected the existence of lumps while processing MRI images of the brain [19]. A simpler way to find lumps consists of applying a guided homotopic thinning procedure to an $N \times N \times N$ cube, using different randomly generated priority functions, until no simple point remains. The following table summarises the outcome of such an experiment, with different values of N and for 10,000 skeletons generated using different random priority functions. We denote by p the proportion of the cases where the result is not a singleton set.

N	10	20	30	40
p	0.0001	0.0249	0.1739	0.5061

Motivated by these practical considerations, two questions arise: is it possible to detect when a thinning procedure gets stuck on a lump, and then, is it possible to find a way towards a globally minimal homotopic skeleton? For performing the latter task, a solution consists of identifying a subset of X which can be removed without changing topology; we call such a subset a *simple set*. Certain classes of simple sets have been studied in the literature dedicated to parallel homotopic thinning algorithms [20,1,12]. In these studies, the considered simple sets are composed exclusively of simple points. In our case, the situation is radically different since a lump relative to K does not contain any simple point outside of K. Then, our problem may be formulated as follows: does there exist a characterisation of certain simple sets composed of non-simple points?

We are indeed interested essentially by simple sets which are minimal, in the sense that they do not strictly include any other simple set, since it is sufficient to detect such sets in order to carry on thinning. Also, we hope that minimal simple sets have a specific structure which could make them easier to analyse.

This paper is dedicated to the study of the simplest ones among such simple sets, called simple pairs, which are those composed of two non-simple points. Our experiments showed us that these minimal simple sets are the ones which are most likely to appear in practical applications, hence the interest in understanding their structure. After proving some properties of simple pairs, we give a characterisation of these sets which allows to detect and remove them when performing homotopic thinning. This paper is self-contained, however the proofs cannot be included due to space limitation. They can be found in [18].

We shall develop this work in the framework of abstract complexes. Abstract complexes have been promoted in particular by V. Kovalevsky [17] in order to provide a sound topological basis for image analysis. In this framework, we retrieve the main notions and results of digital topology, such as the notion of simple point.

2 Cubical complexes

Intuitively, a cubical complex may be thought of as a set of elements having various dimensions (*e.g.* cubes, squares, edges, vertices) glued together accord-

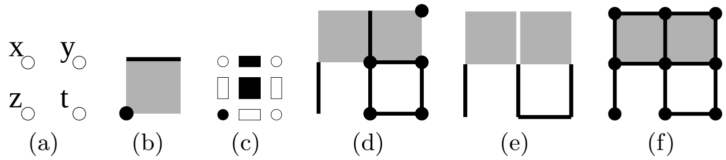


Fig. 2. (a) Four points x, y, z, t of $\mathbb{F}^2$ such that $\{x, y, z, t\}$ is a 2-face. (b,c) Two representations of the set of faces $\{\{x, y, z, t\}, \{x, y\}, \{z\}\}$. (d) A set F of faces in $\mathbb{F}^2$: we see that F is not a complex. (e) The set F^+, composed by the facets of F. (f) The set F^-, *i.e.* the closure of F, which is a complex.

ing to certain rules. In this section, we recall briefly some basic definitions on complexes, see also [5,3,4] for more details. For some illustrations of the notions defined hereafter, the reader may refer to Fig. 2.

Let $\mathbb{Z}$ be the set of integers. We consider the families of sets $\mathbb{F}^1_0$, $\mathbb{F}^1_1$, such that $\mathbb{F}^1_0 = \{\{a\} \mid a \in \mathbb{Z}\}$, $\mathbb{F}^1_1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. A subset f of $\mathbb{Z}^n$ $(n \geq 1)$ which is the Cartesian product of exactly m elements of $\mathbb{F}^1_1$ and $(n - m)$ elements of $\mathbb{F}^1_0$ is called a *face* or an *m-face* of $\mathbb{Z}^n$, m is the *dimension* of f, and we write $\dim(f) = m$.

We denote by $\mathbb{F}^n$ the set composed of all m-faces of $\mathbb{Z}^n$ ($m = 0$ to n). An m-face of $\mathbb{Z}^n$ is called a *point* if $m = 0$, a *(unit) interval* if $m = 1$, a *(unit) square* if $m = 2$, a *(unit) cube* if $m = 3$. In the sequel, we will focus on $\mathbb{F}^3$.

Let f be a face in $\mathbb{F}^3$. We set $\hat{f} = \{g \in \mathbb{F}^3 \mid g \subseteq f\}$, and $\hat{f}^* = \hat{f} \setminus \{f\}$. Any $g \in \hat{f}$ is a *face* of f, and any $g \in \hat{f}^*$ is a *proper face* of f. If F is a finite set of faces of $\mathbb{F}^3$, we write $F^- = \bigcup\{\hat{f} \mid f \in F\}$, F^- is the *closure* of F.

A set F of faces of $\mathbb{F}^3$ is a *cell* or an *m-cell* if there exists an m-face $f \in F$ such that $F = \hat{f}$. The *boundary* of a cell $\hat{f}$ is the set $\hat{f}^*$.

A finite set F of faces of $\mathbb{F}^3$ is a *complex* (in $\mathbb{F}^3$) if for any $f \in F$, we have $\hat{f} \subseteq F$, *i.e.*, if $F = F^-$. Any subset G of a complex F which is also a complex is a *subcomplex* of F. If G is a subcomplex of F, we write $G \preceq F$. If F is a complex in $\mathbb{F}^3$, we also write $F \preceq \mathbb{F}^3$.

A face $f \in F$ is a *facet* of F if there is no $g \in F$ such that $f \in \hat{g}^*$. We denote by F^+ the set composed of all facets of F. Observe that $(F^+)^- = F^-$ and thus, that $(F^+)^- = F$ whenever F is a complex.

The *dimension* of a non-empty complex $F \in \mathbb{F}^3$ is defined by $\dim(F) = \max\{\dim(f) \mid f \in F^+\}$. We say that F is an *m-complex* if $\dim(F) = m$.

Two distinct faces f and g of $\mathbb{F}^3$ are *adjacent* if $f \cap g \neq \emptyset$. Let $F \preceq \mathbb{F}^3$ be a non-empty complex. A sequence $(f_i)_{i=0}^{\ell}$ of faces of F is a *path in F (from f_0 to f_ℓ)* if f_i and f_{i+1} are adjacent, for all $i \in [0, \ell - 1]$. We say that F *is connected* if, for any two faces f, g in F, there is a path from f to g in F. We say that $G \neq \emptyset$ is a *connected component of* F if $G \preceq F$, G is connected and if G is maximal for these two properties (*i.e.*, we have $H = G$ whenever $G \preceq H \preceq F$ and H is connected). We denote by $C[F]$ the set of all the connected components of F. We set $C[\emptyset] = \emptyset$.

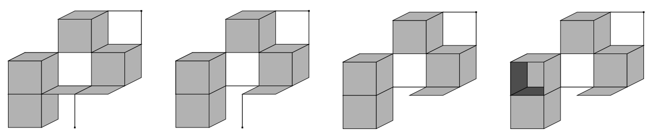
3 Topology preserving operations

Collapse

The collapse, a well-known operation in algebraic topology [13], leads to a notion of homotopy equivalence in discrete spaces, which is the so-called simple homotopy equivalence [7]. To put it briefly, the collapse operation preserves topology.

Let F be a complex in $\mathbb{F}^3$ and let $f \in F^+$. If there exists a face $g \in \hat{f}^*$ such that f is the only face of F which includes g, then we say that the pair (f, g) is a *free pair* for F. If (f, g) is a free pair for F, the complex $F \setminus \{f, g\}$ is an *elementary collapse* of F.

Let F, G be two complexes. We say that F *collapses onto* G if there exists a *collapse sequence from* F *to* G, *i.e.,* a sequence of complexes $\langle F_0, \ldots, F_\ell \rangle$ such that $F_0 = F$, $F_\ell = G$, and F_i is an elementary collapse of F_{i-1}, $i = 1, \ldots, \ell$. Steps of elementary collapse of a 3-D complex are illustrated in Fig. 3.



(a) (b) (c) (d)

Fig. 3. (a) A complex $F \preceq \mathbb{F}^3$. (b), (c), (d) Three steps of elementary collapse of F.

Let F, G be two complexes. Let H such that $F \cap G \preceq H \preceq G$, and let $f, g \in H \setminus F$. The pair (f, g) is a free pair for $F \cup H$ if and only if (f, g) is a free pair for H. Thus, by induction, we have the following proposition.

Proposition 1 ([2]) *Let* $F, G \preceq \mathbb{F}^3$. *The complex* $F \cup G$ *collapses onto* F *if and only if* G *collapses onto* $F \cap G$.

Topological invariants

Let F be a complex in $\mathbb{F}^3$, and let us denote by n_i the number of i-faces of F, $i = 0, \ldots, 3$. The *Euler characteristic* of F, written $\chi(F)$, is defined by $\chi(F) = n_0 - n_1 + n_2 - n_3$. The Euler characteristic is a well-known topological invariant, in particular, it is easy to see that the collapse operation preserves it. This invariant will play an essential role in the proofs of this paper.

Let $F, G \preceq \mathbb{F}^3$. A fundamental and well-known property of the Euler characteristic, analog to the so-called inclusion-exclusion principle in set theory, is the following: $\chi(F \cup G) = \chi(F) + \chi(G) - \chi(F \cap G)$.

The Euler-Poincaré formula shows a deep link between the Euler characteristic and the Betti numbers, which are topological invariants defined from the homology groups of a complex. Intuitively[1], the Betti numbers b_0, b_1, b_2 corre-

[1] An introduction to homology theory can be found *e.g.* in [13].

spond respectively to the number of connected components, tunnels and cavities of F. The Euler-Poincaré formula, in the case of a complex F in $\mathbb{F}^3$, states that $\chi(F) = b_0 - b_1 + b_2$. Betti numbers are also preserved by collapse.

Simplicity

Intuitively, a part of a complex F is called simple if it can be "removed" from F while preserving topology. We recall here a definition of simplicity (see [2]) based on the collapse operation, which can be seen as a discrete counterpart of the one given by T.Y. Kong [15].

Definition 2 *Let $G \preceq F \preceq \mathbb{F}^3$. We set $F \oslash G = (F^+ \setminus G^+)^-$. The set $F \oslash G$ is a complex which is the* detachment *of G from F.*
We say that G is simple for F *if F collapses onto $F \oslash G$. Such a subcomplex G is called a* simple subcomplex of F *or a* simple set for F.

It has to be noticed that this definition of simple set is different (and more general) than the one proposed in [12,14].

Let $G \preceq F \preceq \mathbb{F}^3$. The *attachment* of G for F is the complex defined by $Att(G, F) = G \cap (F \oslash G)$. This notion of attachment leads to a local characterisation of simple sets: Prop. 3 is a special case of Prop. 1 as $(F \oslash G) \cup G = F$.

Proposition 3 *Let $G \preceq F \preceq \mathbb{F}^3$. The complex G is simple for F if and only if G collapses onto $Att(G, F)$.*

4 Minimal simple pairs in $\mathbb{F}^3$

In the image processing literature, a digital image is often considered as a set of pixels in 2-D or voxels in 3-D. A voxel is an elementary cube, thus an easy correspondence can be made between this classical view and the framework of cubical complexes. In the sequel of the paper, we call *voxel* any 3-cell. If a complex $F \preceq \mathbb{F}^3$ is a union of voxels, we write $F \sqsubseteq \mathbb{F}^3$. If $F, G \sqsubseteq \mathbb{F}^3$ and $G \preceq F$, then we write $G \sqsubseteq F$. From now on, we consider only complexes which are unions of voxels.

Notice that, if $F \sqsubseteq \mathbb{F}^3$ and if $\hat{f}$ is a voxel of F, then $F \oslash \hat{f} \sqsubseteq \mathbb{F}^3$. There is indeed an equivalence between the operation on complexes that consists of removing (by detachment) a simple voxel, and the removal of a 26-simple voxel in the framework of digital topology (see [14,4]).

As discussed in the introduction, the minimal simple sets which are most likely to appear in thinning processes are those which are composed of only two voxels. In this paper, we will concentrate on this particular - but very frequent - case, and provide a definition, some properties and a characterisation of these sets.

Definition 4 *Let $G \sqsubseteq F, G \neq \emptyset$. The subcomplex G is a* minimal simple set (for F) *if G is a simple set for F and G is minimal with respect to the relation $\sqsubseteq$*

*(*i.e. *$H = G$ whenever $H \sqsubseteq G$ and H is a non-empty simple set for F).*
Let P be a minimal simple set for F which is composed of two voxels. Then, we call P a minimal simple pair, or MSP (for *F).*

Observe that, if a voxel is a simple cell for F, then it is also a (minimal) simple set for F. Thus, any minimal simple set which contains strictly more than one voxel cannot contain any simple voxel. In particular, if P is a simple set which contains only two voxels, then P is a MSP if and only if it does not contain any simple voxel.

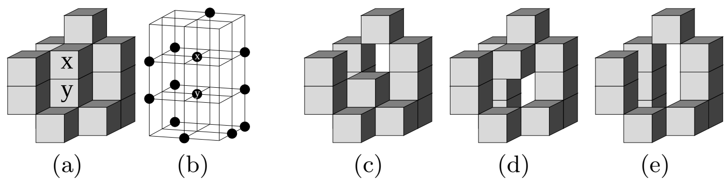


Fig. 4. Example of a MSP (voxels x and y). (a), (b): Two representations of the same complex F. (c), (d), (e): Effect of removing either x, y or both (see text).

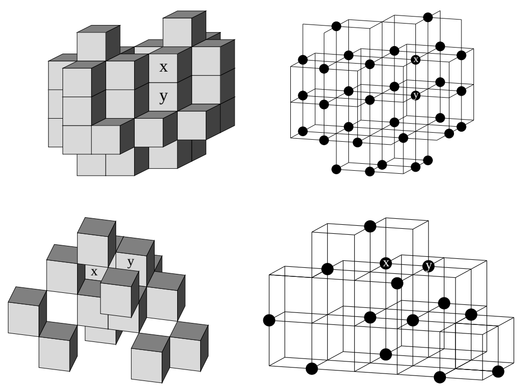


Fig. 5. Left: two complexes composed of non-simple voxels. Right: another representation of these complexes. The subset $\{x, y\}$ is a MSP for both of them (the removal of $\{x, y\}$ will not alter their topology).

Before beginning the study of MSPs (next section), let us show an example of such a configuration. Consider the complex F depicted in Fig. 4a. Another representation of this object is shown in Fig. 4b, where each cube (voxel) is represented by a black dot. It can easily be seen that the complex F is connected and has no cavity and no tunnel; furthermore it can be reduced to a single voxel by iterative deletion of simple voxels. Let us now concentrate on the set formed by the two voxels x and y.

In Fig. 4c, we can see that removing x from F creates a tunnel. Thus x is not a simple voxel. The same can be said about y (see Fig. 4d). But if both x and y are removed (see Fig. 4e), then we see that we obtain a complex G which has no tunnel. It is easily verified that the union of the cells x and y is in fact a simple subcomplex of F, so that it is a MSP for F.

Of course, the complex F of Fig. 4a is not a lump since it contains simple voxels (on its border). In Fig. 5 (1st row), we show that the same configuration can appear in a complex which has no simple voxel but is however topologically equivalent to a single voxel. This lump can be homotopically reduced by deletion of the simple pair $\{x, y\}$. The obtained result could then be further reduced to a singleton set by iterative simple voxel removal. Notice that this complex (generated by randomised homotopic thinning from a 5 voxel-width cube) is made of only 32 voxels.

There exist examples containing less points: the smallest one we found until now is composed of only 14 voxels and has some tunnels (see Fig. 5, 2nd row). We conjecture that 14 is the smallest size for a lump containing a MSP.

We conclude this section by quoting a characterisation of 3-D simple voxels proposed by Kong in [15], which is equivalent to the following theorem for subcomplexes of $\mathbb{F}^3$; this characterisation will be used in the next section. Remind that $|C[X]|$ denotes the number of connected components of X.

Theorem 5 (Adapted from Kong [15]) *Let $F \sqsubseteq \mathbb{F}^3$. Let $g \in F^+$. Then $\hat{g}$ is a simple voxel for F if and only if $|C[Att(\hat{g}, F)]| = 1$ and $\chi(Att(\hat{g}, F)) = 1$.*

5 Some properties of minimal simple pairs

We are now ready to state some properties about the structure of MSPs: first of all, a simple set need not be connected, but any MSP is indeed connected.

Proposition 6 *Let $P \sqsubseteq F$ be a MSP for F. Then:*

$$|C[P]| = 1.$$

As discussed before, the voxels constituting a MSP cannot be simple voxels. Intuitively, the attachment of a non-simple voxel $\hat{f}$ can either: i) be empty (isolated voxel), ii) be equal to the boundary of $\hat{f}$ (interior voxel), iii) be disconnected, iv) have at least one tunnel. Notice that iii) and iv) are not exclusive, the attachment of a non-simple voxel can both be disconnected and contain tunnels.

We will see that some of these cases cannot appear in a MSP. First, we prove that i) and iii) cannot hold for such a voxel, *i.e.*, the attachment of a voxel in a MSP is non-empty and connected.

Proposition 7 *Let $P \sqsubseteq F$ be a MSP for F. Then:*

$$\forall g \in P^+, |C[Att(\hat{g}, F)]| = 1.$$

Then, with the next proposition, we show that ii) cannot hold, hence, the attachment to F of any voxel g in a MSP has no cavity.

Proposition 8 *Let $P \sqsubseteq F$ be a MSP for F. Then:*

$$\forall g \in P^+, Att(\hat{g}, F) \neq \hat{g}^*.$$

Recall that, according to the Euler-Poincaré formula, $\chi(Att(\hat{g}, F)) = b_0 - b_1 + b_2$, where b_0 (resp. b_2) is the number of connected components (resp. cavities) of $Att(\hat{g}, F)$. From the two previous propositions, we have $b_0 = 1$ and $b_2 = 0$. The Betti number b_1, which represents the number of tunnels, is positive. Thus, we have $\chi(Att(\hat{g}, F)) = 1 - b_1 \leq 1$. But from Theorem 5 and Prop. 7 we must have $\chi(Att(\hat{g}, F)) \neq 1$, otherwise g would be a simple voxel. This proves the following proposition, which (with Prop. 7 and Prop. 8) implies that the attachment to F of any voxel in a MSP has at least one tunnel.

Proposition 9 *Let $P \sqsubseteq F$ be a MSP for F. Then:*

$$\forall g \in P^+, \chi(Att(\hat{g}, F)) \leq 0.$$

From Prop. 6, we know that a MSP is necessarily connected. The following proposition tells us more about the intersection of the two voxels which compose any MSP.

Proposition 10 *Let $P \sqsubseteq F$ be a MSP for F, and let g_1, g_2 be the two voxels of P. Then, $g_1 \cap g_2$ is a 2-face.*

This proposition is indeed an easy consequence of the following lemma: it may be seen that Lemma 11 implies that the intersection of $Att(P, F)$ with $g_1 \cap g_2$ has at least three connected components. This is possible only when $\dim(g_1 \cap g_2) = 2$.

Lemma 11 *Let $P \sqsubseteq F$ be a MSP for F, and let g_1, g_2 be the two voxels of P. Then, $\chi(Att(P, F) \cap \hat{g_1} \cap \hat{g_2}) \geq 3$.*

To illustrate the above properties, let us consider the attachment of the pair $P = \{x, y\}$ of Fig. 4a, which is displayed in Fig. 6 (left), and the attachments of x and y displayed in Fig. 6 (middle and right, respectively). We can see in particular that the intersection of $Att(P, F)$ with $x \cap y$ is indeed composed of three connected components (the 0-cells u, v and w), as implied by Lemma 11.

The two following propositions are necessary conditions for a MSP, which are similar to the conditions of Theorem 5 which characterise simple voxels.

From Prop. 3, P collapses onto $Att(P, F)$ whenever P is a MSP. We have $\chi(P) = 1$, and from Prop. 6, $|C[P]| = 1$. Since collapse preserves the number of connected components and the Euler characteristic, we have the following.

Proposition 12 *Let $P \sqsubseteq F$ be a MSP for F. Then:*

$$|C[Att(P, F)]| = 1.$$

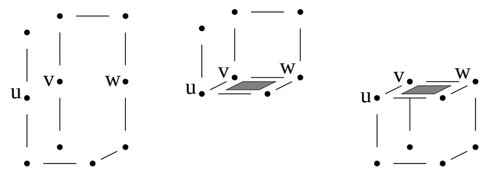


Fig. 6. Attachments of configurations of Fig. 4. From left to right: attachment of $\{x, y\}$, attachment of x, attachment of y.

Proposition 13 *Let $P \sqsubseteq F$ be a MSP for F. Then:*

$$\chi(Att(P, F)) = 1.$$

Finally, we give a characterisation of MSPs, which summarises and extends the properties shown before.

Proposition 14 *Let $P \sqsubseteq F$ be a pair. Then P is a MSP for F if and only if all the following conditions hold:*

$$\text{the intersection of the two voxels of } P \text{ is a 2-face}, \tag{1}$$
$$\forall g \in P^+, |C[Att(\hat{g}, F)]| = 1, \tag{2}$$
$$\forall g \in P^+, \chi(Att(\hat{g}, F)) \leq 0, \tag{3}$$
$$|C[Att(P, F)]| = 1, \tag{4}$$
$$\chi(Att(P, F)) = 1. \tag{5}$$

Remark 15 *Conditions (1), (3), (4), and (5) are sufficient to characterise a MSP, since condition (2) may be deduced from (1), (3), (4). Moreover, if P is a pair of non-simple voxels, then P is a MSP for F if and only if conditions (4) and (5) both hold. We retrieve a characterisation similar to Theorem 5.*

6 Conclusion

The notion of simple voxel (or simple point), which is commonly considered for topology-preserving thinning, is sometimes not sufficient to obtain reduced objects being globally minimal. The detection of MSPs (and more generally of minimal simple complexes) can then enable to improve the thinning procedures by "breaking" specific objects such as the ones studied here.

For example, let us consider again the experiment described in the introduction. Among 10,000 objects obtained by applying a homotopic thinning procedure guided by a random priority function to a $20 \times 20 \times 20$ full cube, we found 249 lumps. In 212 of these 249 cases, further thinning was made possible by the detection of a MSP. In 203 of these 212 cases, it has been possible to continue the thinning process until obtaining a single voxel.

It has to be noticed that the search of MSPs in a complex $F \sqsubseteq \mathbb{F}^3$ does not present an algorithmic complexity higher than the search of simple voxels (both being linear with respect to the number of facets of the processed complex). Consequently, it is possible to create new thinning procedures based on the detachment of both simple voxels and pairs and whose runtimes have the same order of growth as the runtimes of thinning procedures that are based only on simple voxels. Such new algorithms would be able to produce skeletons having less points than standard ones.

A Appendix: Simple equivalence and MSPs

We define here the notion of lump, informally introduced in Section 1.

Definition 16 *Let $F, G \sqsubseteq \mathbb{F}^3$. We say that F and G are* simple-equivalent *if there exists a sequence of complexes $\langle F_0, \ldots, F_\ell \rangle$ such that $F_0 = F$, $F_\ell = G$, and for any $i \in \{1, \ldots, \ell\}$, we have either*
i) $F_i = F_{i-1} \oslash x_i$, where x_i is a voxel which is simple for F_{i-1} ; or
ii) $F_{i-1} = F_i \oslash x_i$, where x_i is a voxel which is simple for F_i.

Definition 17 *Let $G \sqsubseteq F \sqsubseteq \mathbb{F}^3$, such that F and G are simple-equivalent. If $F \neq G$ and F does not contain any simple voxel outside of G, then we say that F is a* lump relative to *G, or simply a* lump.

For example, the Bing's house of Fig. 1 is a lump (relative to any one of its voxels), which is composed of 135 voxels. Another example of lump, much simpler, if given in Fig. 7 (left) (see in [18], Appendix C, some steps of a sequence which shows that Fig. 7 (left) and Fig. 7 (right) are simple-equivalent).

Remark 18 *The preceding example invites us to consider a notion based on simple-equivalence, which is more general than the one of simple set. A subcomplex $G \sqsubseteq F$ is called* SE-simple for *F if F and $F \oslash G$ are simple-equivalent. For example, the voxel x in the complex F of Fig. 7 (left) is SE-simple for F, although it is not a simple voxel for F (this kind of configuration has been analysed in [11]). Of course, any simple set is SE-simple, and the preceding example proves that the converse is not true in general. However, it is not possible to characterise locally, in the manner of Prop. 3, a voxel or a set which is SE-simple. This is why we use Def. 2 as the definition of a simple set.*

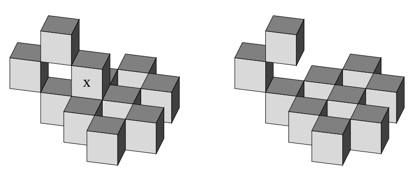


Fig. 7. On the left, the smallest lump found so far. It contains no simple voxel, and is simple-equivalent to the complex on the right, made of 10 voxels. Both objects have three tunnels.

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