

On a (β, q) -generalized Fisher information and inequalities involving q -Gaussian distributions

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(Dated: Submitted January 25 2012, Revised May 12, 2012)

In the present paper, we would like to draw attention to a possible generalized Fisher information that fits well in the formalism of nonextensive thermostatics. This generalized Fisher information is defined for densities on \mathbb{R}^n . Just as the maximum Rényi or Tsallis entropy subject to an elliptic moment constraint is a generalized q -Gaussian, we show that the minimization of the generalized Fisher information also leads a generalized q -Gaussian. This yields a generalized Cramér-Rao inequality. In addition, we show that the generalized Fisher information naturally pops up in a simple inequality that links the generalized entropies, the generalized Fisher information and an elliptic moment. Finally, we give an extended Stam inequality. In this series of results, the extremal functions are the generalized q -Gaussians. Thus, these results complement the classical characterization of the generalized q -Gaussian and introduce a generalized Fisher information as a new information measure in nonextensive thermostatics.

PACS numbers: 02.50.-r, 05.90.+m, 89.70

Keywords: Generalized entropies, Fisher information, q -Gaussian distributions, inequalities.

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I. INTRODUCTION

The Gaussian distribution has a central role with respect to standard information measures. For instance, it is well known that the Gaussian distribution maximizes the entropy over all distributions with the same variance; see Dembo et al.¹¹, Lemma 5. Similarly, the Stam³⁶ inequality shows that the minimum of the Fisher information over all distributions with a given entropy also occurs for the Gaussian distribution. Finally, the Cramér-Rao inequality, e.g. Dembo et al.¹¹, Theorem 20, shows that the minimum of the Fisher information over all distributions with a given variance is attained for the Gaussian distribution.

It is thus natural to inquire for similar results for extended entropies and an associated generalized Fisher information. In the context of the nonextensive thermostatics introduced by Tsallis^{37,38}, the role devoted to the standard Gaussian distribution is extended to generalized q -Gaussian distributions, see e.g. Lutz²⁶, Schwämmle et al.³⁵, Vignat and Plastino³⁹, Ohara and Wada³¹. The generalized q -Gaussians are also explicit extremal functions of Sobolev, log-Sobolev or Gagliardo–Nirenberg inequalities on \mathbb{R}^n , with $n \geq 2$, see Del Pino and Dolbeault^{9,10}. This family of distributions will precisely achieve the equality case in all the information inequalities presented in the sequel.

Definition 1. Let x be a random vector of \mathbb{R}^n . For $\alpha \in (0, \infty)$, γ a real positive parameter and $q > (n - \alpha)/n$, the generalized q -Gaussian with scale parameter γ has the radially symmetric probability density

$$G_\gamma(x) = \begin{cases} \frac{1}{Z(\gamma)} (1 - (q - 1) \gamma |x|^\alpha)_+^{\frac{1}{q-1}} & \text{for } q \neq 1 \\ \frac{1}{Z(\gamma)} \exp(-\gamma |x|^\alpha) & \text{if } q = 1 \end{cases} \quad (1)$$

where $|x|$ denotes the standard euclidean norm, where we use the notation $(x)_+ = \max\{x, 0\}$, and where $Z(\gamma)$ is the partition function such that $G_\gamma(x)$ integrates to one. Its expression is given in (A2).

For $q > 1$, the density has a compact support, while for $q \leq 1$ it is defined on the whole \mathbb{R}^n and behaves as a power distribution for $|x| \rightarrow \infty$. Notice that the name generalized Gaussian is sometimes restricted to the case $q = 1$ above. In the following, we will simply note G , rather than G_1 , the generalized q -Gaussian obtained with $\gamma = 1$. The general expressions of the main information measures attached to the generalized Gaussians are derived in Appendix A.

Let us recall that if x is a vector of \mathbb{R}^n with probability density $f(x)$ with respect to the Lebesgue measure, then the information generating function introduced by Golomb¹⁶ is defined by

$$M_q[f] = \int f(x)^q dx, \quad (2)$$

for $q \geq 0$. Associated with this information generating function, the Rényi entropy is given by

$$H_q[f] = \frac{1}{1-q} \log M_q[f] \quad (3)$$

while the Tsallis entropy is given by

$$S_q[f] = \frac{1}{1-q} (1 - M_q[f]). \quad (4)$$

Finally, we will also call “entropy power” of order q , or q -entropy power, the quantity

$$N_q[f] = M_q[f]^{\frac{1}{1-q}} = \exp(H_q[f]) = \left(\int f^q dx \right)^{\frac{1}{1-q}}, \quad (5)$$

for $q \neq 1$. For $q = 1$, we let $N_q[f] = \exp(H_q[f])$. Note that the entropy power is usually defined as the square of the quantity above, with an additional factor. We will also use the definition of the elliptic moment of order α , given by

$$m_\alpha[f] = \int |x|^\alpha f(x) dx, \quad (6)$$

where $|x|$ denotes the euclidean norm of x .

In the context of nonextensive thermostatics, the study of the role of Fisher information, or the definition of an extension has attracted some efforts. Several authors have introduced generalized versions of the Fisher information and derived the corresponding Cramér-Rao bounds. Among these contributions we note the series of papers Casas et al.⁵, Chimento et al.⁶, Pennini et al.^{32,33} where the proposed extended Fisher information involves a power f^q of the original distribution f , or a normalized version which is the escort distribution. We shall point out here the early contribution of P. Hammad¹⁷ that essentially introduced the previous escort-Fisher information. A modified version, that also involves escort distributions, is considered by Naudts^{29,30}. A recent contribution in the nonextensive literature and in this journal is the proposal of Furuichi^{13,14}, where the logarithm is replaced by a deformed one and a Cramér-Rao type inequality is given.

In the present paper, we present an extension of the Fisher information measure, the (β, q) -Fisher information $I_{\beta,q}[f]$ that depends on an entropic index q and on a parameter

β . This information has been introduced by Lutwak et al.²³. It reduces to the standard Fisher information in the case $q = 1, \beta = 2$. We look for the probability density function with minimum generalized Fisher information, among all probability density functions with a given elliptic moment, let us say $m_\alpha[f] = m$:

$$I_{\beta,q}(m) = \inf_f \{I_{\beta,q}[f] : f(x) \geq 0, \int f(x)dx=1 \text{ and } m_\alpha[f] = m\}. \quad (7)$$

It will turn out that the extremal distribution is precisely the generalized q -Gaussian with the prescribed elliptic moment. Actually, there is a more precise statement of the result in the form of a Cramér-Rao inequality that links the Fisher information and the elliptic moment $m_\alpha[f]$. It is shown in section IV that for α and β Hölder conjugates of each other, with $\alpha \geq 1$, then

$$I_{\beta,q}[f]^{\frac{1}{\beta\lambda}} m_\alpha[f]^{\frac{1}{\alpha}} \geq I_{\beta,q}[G]^{\frac{1}{\beta\lambda}} m_\alpha[G]^{\frac{1}{\alpha}} \quad (8)$$

with $\lambda = n(q-1) + 1$, and where the equality is obtained for any generalized q -Gaussian distribution $f = G_\gamma$. This inequality means in particular that the minimum of the generalized Fisher information subject to an elliptic moment constraint is reached by a generalized Gaussian. This complements the classical characterization of the generalized q -Gaussian that are found as the maximum Rényi or Tsallis entropy distributions with a given moment. Incidentally, we will also find, from a general sharp Gagliardo-Nirenberg inequality, that the generalized entropy power and Fisher information are linked by an inequality that generalizes the Stam inequality

$$N_q[f] I_{\beta,q}[f]^{\frac{n}{\beta\lambda}} \geq N_q[G] I_{\beta,q}[G]^{\frac{n}{\beta\lambda}}, \quad (9)$$

again with $\lambda = n(q-1) + 1$.

Actually, the inequalities (8) and (9) have been presented in the monodimensional case in the remarkable paper by Lutwak et al.²³ whose reading has deeply influenced this work. Beyond the highlighting on the result in the nonextensive statistics context, a contribution of the present paper is the extension of the definitions and results in the multidimensional case. Furthermore, we show in section III, that the expression of the generalized Fisher information naturally pops up from a simple application of Hölder's inequality, which leads to an inequality that links the generalized Fisher information, the Rényi entropy power and the elliptic moment. We show that this inequality, which is saturated by the generalized q -Gaussian, immediately suggests the Cramér-Rao inequality (8). This Cramér-Rao inequality is derived in section IV with the help of the generalized Stam inequality (9).

II. ON THE GENERALIZED (β, q) -FISHER INFORMATION

We begin by the definition of the generalized (β, q) -Fisher information and its main consequences.

Definition 2. Let $f(x)$ be a probability density function defined over a subset Ω of \mathbb{R}^n . Let $|x|$ denote the euclidean norm of x and ∇f the gradient of function $f(x)$ with respect to x . If $f(x)$ is continuously differentiable over Ω , then for $q \geq 0$, $\beta > 1$, the generalized (β, q) -Fisher information is defined by

$$I_{\beta,q}[f] = \int_{\Omega} f(x)^{\beta(q-1)+1} \left(\frac{|\nabla f(x)|}{f(x)} \right)^{\beta} dx. \quad (10)$$

The standard Fisher information ($q = 1$, $\beta = 2$), see for instance Cohen⁷, Huber and Ronchetti¹⁹, Plastino et al.³⁴ is a convex functional of the probability density f . It was shown by Boekee³ that this is also true for the generalized Fisher information with $q = 1$ and any $\beta > 1$. It is also easy to check that this is still true in the case $\beta(q-1)+1 = \beta$. Unfortunately, the general characterization of the convexity or pseudo-convexity properties of the generalized Fisher information remains an open problem.

A radially symmetric function $f(x)$ only depends on the euclidean distance $r = |x|$ of the argument from the origin. Hence, in the case of radially symmetric function defined on a symmetric domain Ω , we have $f(x) = f_r(|x|) = f_r(r)$, where f_r is an univariate probability density defined on a subset Ω_r of \mathbb{R}^+ . With these notations, we simply have $|\nabla f(x)| = \left| \frac{df_r(r)}{dr} \right|$. In polar coordinates, with $r = |x|$, $dx = r^{n-1} dr du$ and $\int du = n\omega_n$, where ω_n is the volume of the n -dimensional unit ball, the expression of the generalized Fisher information becomes

$$I_{\beta,q}[f] = n\omega_n \int_{\Omega_r} r^{n-1} f_r(r)^{\beta(q-1)+1} \left| \frac{f'_r(r)}{f_r(r)} \right|^{\beta} dr.$$

In the following, it will also be convenient to use the simple transformation $f(x) = u(x)^k$, with $k = \beta / (\beta(q-1)+1)$. Indeed, with this notation and taking into account that $|\nabla f| = |k|u^{k-1}|\nabla u|$, the generalized Fisher information reduces to

$$I_{\beta,q}[f] = |k|^{\beta} \int_{\Omega} |\nabla u(x)|^{\beta} dx, \quad (11)$$

which corresponds to the β -Dirichlet energy of $u(x)$. In the case $k = 2$, the function $u(x)$ is analog to a wave function in quantum mechanics.

The variational problem (7) can be restated as follows:

$$I_{\beta,q}(m) = \inf_u \left\{ \int_{\Omega} |\nabla u(x)|^{\beta} dx : \int_{\Omega} u(x)^k dx = 1 \text{ and } m = \int_{\Omega} |x|^{\alpha} u(x)^k dx \right\}. \quad (12)$$

The Lagrangian associated with the problem (12) is

$$L(u; a, b) = \int_{\Omega} |\nabla u(x)|^{\beta} dx + a \int_{\Omega} u(x)^k dx + b \int_{\Omega} |x|^{\alpha} u(x)^k dx \quad (13)$$

where a and b are the Lagrange parameters associated with the two constraints. It is remarkable to note that the same kind of formulation has been introduced in the book by Frieden¹², and solutions derived in the cases $k = 2$ and $k = 1$, with the mention of the physical interest of the latter case. The Euler-Lagrange equation, see Brunt⁴, corresponding to the variational problem (12) is

$$\operatorname{div} (|\nabla u(x)|^{\beta-2} \nabla u(x)) - \frac{k}{\beta} (a + b|x|^{\alpha}) u(x)^{k-1} = 0, \quad (14)$$

It has the form of a p -Laplace equation as in Lindqvist²², here with $p = \beta$,

$$\Delta_{\beta} u(x) - \frac{k}{\beta} (a + b|x|^{\alpha}) u(x)^{k-1} = 0,$$

where Δ_{β} denotes the β -Laplacian operator. Observe that in the special case $k = 2$, (14) is analogous to the Schrödinger equation for the quantum oscillator in a relativistic setting (Klein-Gordon equation). In such case, it is known that the ground state is a normal distribution. In the radial case, with $r = |x|$ and $u(x) = u_r(r)$, the nonlinear second order differential equation reduces to

$$(r^{n-1} u_r'(r)^{\beta-1})' - \frac{k}{\beta} r^{n-1} (a + br^{\alpha}) u_r(r)^{k-1} = 0. \quad (15)$$

In our context, it is thus natural to suspect that the generalized q -Gaussian could be a solution. Actually, it is easy to check that if G_{γ} is the generalized q -Gaussian (1), then $u(x) = G_{\gamma}(x)^{\frac{1}{k}}$ is indeed a solution of this equation, with the following relations between the parameters:

$$a = -An, \quad b = A(1 + n(q-1))\gamma \quad \text{where } A = \left(\frac{\beta}{k}\right)^{\beta} \left(\frac{\gamma}{\beta-1}\right)^{\beta-1} Z(\gamma)^{k-\beta}. \quad (16)$$

It will be shown in the following that the generalized Gaussian is not only a possible solution, but the actual optimum solution of the problem.

Let us finally note that in the case $\beta = 2$, the nonlinear differential equation (15) is an instance of the generalized Emden-Fowler equation $u''(x) + h(x) u(x)^\gamma = 0$, where $h(x)$ is a given function. This kind of equations arise in studies of gaseous dynamics in astrophysics, in certain problems in fluid mechanics and pseudoplastic flow, see Nachman and Callegari²⁷, as well as in some reaction-diffusion processes, as mentioned by Covei⁸.

In the radial case, the minimum Fisher information can be written in terms of the constraints and of the Lagrange parameters associated with these constraints.

Proposition 1. *Let Ω be the n -dimensional ball of radius R , possibly infinite, centered on the origin. Among all radial densities $f = u^k$ defined on Ω and such that $f_r(R) = 0$, the minimum Fisher information in (12) can be expressed in terms of the constraints and the Lagrange multipliers as*

$$\frac{1}{|k|^\beta} I_{\beta,q}(m) = -\frac{k}{\beta} (a + bm). \quad (17)$$

Proof. By integration by parts,

$$n \omega_n \int_0^R r^{n-1} (u'_r(r)^\beta) dr = n \omega_n \int_0^R r^{n-1} (u'_r(r)^{\beta-1}) u'_r(r) dx \quad (18)$$

$$= n \omega_n \left[(r^{n-1} u'_r(x)^{\beta-1}) u_r(x) \right]_0^R - n \omega_n \int_0^R u_r(r) (r^{n-1} u'_r(r)^{\beta-1})' dx \quad (19)$$

Using the boundary condition and the differential equation (15), we then obtain

$$n \omega_n \int_0^R r^{n-1} (u'_r(r)^\beta) dr = -n \omega_n \frac{k}{\beta} \int_0^R r^{n-1} (a + br^\alpha) u_r(r)^k dx.$$

Taking into account the values of the constraints, this reduces to (17). \square

It can be noted that the Lagrange parameters are actually (complicated) functions of the constraints, so that the right hand side of (17) is not a simple affine function in m .

III. A FISHER-MOMENT-ENTROPY INEQUALITY AND THE EXTENDED CRAMÉR-RAO INEQUALITY

We show here that the generalized Fisher information naturally pops up in a simple inequality that links this generalized Fisher information, the information generating function and the elliptic moment of order α . Then, we show that this inequality suggests the general Cramér-Rao inequality (8).

Theorem 1. Let $f(x) = f_r(|x|)$ a radially symmetric probability density on the n -dimensional ball Ω of radius R , possibly infinite, centered on the origin. Assume that the density is absolutely continuous and such that $\lim_{r \rightarrow R} r^n f_r(r)^q = 0$. Let also $\alpha \geq 1$ and β be its Hölder conjugate. Then, for $q > n/(n + \alpha)$ and provided that the involved information measures are finite, we have

$$I_{\beta,q}[f]^{\frac{1}{\beta}} m_{\alpha}[f]^{\frac{1}{\alpha}} \geq \frac{n}{q} M_q[f] \quad (20)$$

with equality if and only if f is a generalized Gaussian $f = G_{\gamma}$ for any $\gamma > 0$.

Proof. Let us consider the information generating function $M_q[f]$. Since the density is assumed radially symmetric, the use of polar coordinates reduces the computation to the evaluation of an univariate integral. By integration by parts, we obtain

$$M_q[f] = \int_{\Omega} f(x)^q dx = n\omega_n \int_0^R r^{n-1} f_r(r)^q dr = n\omega_n \left[\frac{r^n}{n} f_r(r)^q \right]_0^R - n\omega_n q \int_0^R \frac{r^n}{n} f'_r(r) f_r(r)^{q-1} dr \quad (21)$$

$$= n\omega_n \frac{q}{n} \int_0^R [r] \left[-\frac{f'_r(r)}{f_r(r)} f_r(r)^{(q-1)} \right] r^{n-1} f_r(r) dr \quad (22)$$

where we used the fact that $\lim_{r \rightarrow R} r^n f_r(r)^q = 0$. We can then use the fact that $|\int g(x) dx| \leq \int |g(x)| dx$ with equality iff $g(x) \geq 0$, and the Hölder inequality which states that

$$\left(\int |u(x)|^{\alpha} w(x) dx \right)^{\frac{1}{\alpha}} \left(\int |v(x)|^{\beta} w(x) dx \right)^{\frac{1}{\beta}} \geq \int |u(x)v(x)| w(x) dx$$

with $\alpha^{-1} + \beta^{-1} = 1$, $\alpha > 1$ and where $w(x) \geq 0$ is a weight function. The equality case then occurs iff $|u(x)|^{\alpha} = k|v(x)|^{\beta}$, with $k > 0$.

Let us now apply these inequalities to the remaining integral in (22), with $g(r) = u(r)v(r)w(r)$, $u(r) = r$, $v(r) = -\frac{f'_r(r)}{f_r(r)} f_r(r)^{(q-1)}$, and $w(r) = r^{n-1} f_r(r)$. Immediately, we obtain

$$\left(n\omega_n \int_0^R \left| \frac{f'_r(r)}{f_r(r)} \right|^{\beta} f_r(r)^{\beta(q-1)} r^{n-1} f_r(r) dr \right)^{\frac{1}{\beta}} \left(n\omega_n \int_0^R r^{\alpha} r^{n-1} f_r(r) dr \right)^{\frac{1}{\alpha}} \geq \frac{n}{q} M_q[f] \quad (23)$$

which is inequality (20) in polar coordinates.

The conditions for equality give here $f'_r(r) < 0$ on the one hand, and $r^{\alpha} = k \left| \frac{f'_r(r)}{f_r(r)} \right|^{\beta} f_r(r)^{\beta(q-1)}$, $k > 0$, on the other hand. Finally, using the fact that $\alpha/\beta = \alpha - 1$, the first-order nonlinear differential equation reduces to

$$r^{\alpha-1} f_r(r)^{2-q} = -K f'_r(r). \quad (24)$$

Then, it is easy to check that the unique normalized solution of the differential equation (24) is nothing but the generalized Gaussian (1). \square

We shall mention that in the monodimensional case, the radial symmetry hypothesis is not necessary, as the integration can be achieved on the real line, and the previous result holds for all probability densities. In the case $q = 1$, the information generating function in the right side of (20) equals to 1, so that the inequality reduces to $I_{\beta,1}[f]^{\frac{1}{\beta}} m_{\alpha}[f]^{\frac{1}{\alpha}} \geq 1$ which is a Cramér-Rao inequality that has been exhibited by Boeke³.

As a direct consequence, we obtain a possible derivation of the more general Cramér-Rao inequality (8). The idea is to lower bound the right side of (20). Let us first consider the case $q > 1$. In this case, $M_q[f]$ is a convex functional, and therefore has a single minimizer among all densities with a given moment. It is well known that this minimizer is a generalized q -Gaussian $G_{\theta}(x)$ with the same moment:

$$M_q[f] \geq \inf_{p/m_{\alpha,1}[p]=m_{\alpha,1}[f]} M_q[p] = M_q[G_{\theta}].$$

Therefore, in the case $q > 1$ the inequality (20) also yields

$$I_{\beta,q}[f]^{\frac{1}{\beta}} m_{\alpha,1}[f]^{\frac{1}{\alpha}} \geq \frac{n}{q} M_q[G_{\theta}], \quad (25)$$

with equality if and only if $f = G_{\theta}$. Of course, the right hand side term can also be written $I_{\beta,q}[G_{\theta}]^{\frac{1}{\beta}} m_{\alpha,1}[G_{\theta}]^{\frac{1}{\alpha}}$, or $I_{\beta,q}[G_{\theta}]^{\frac{1}{\beta}} m_{\alpha,1}[G_{\theta}]^{\frac{\lambda}{\alpha}} m_{\alpha,1}[G_{\theta}]^{\frac{1-\lambda}{\alpha}}$, with $\lambda = (n(q-1) + 1)$. Therefore, the inequality (25) becomes

$$I_{\beta,q}[f]^{\frac{1}{\beta}} m_{\alpha,1}[f]^{\frac{1}{\alpha}} m_{\alpha,1}[G_{\theta}]^{-\frac{1-\lambda}{\alpha}} \geq I_{\beta,q}[G_{\theta}]^{\frac{1}{\beta}} m_{\alpha,1}[G_{\theta}]^{\frac{\lambda}{\alpha}}.$$

Since the optimum distribution G_{θ} is such that $m_{\alpha,1}[f] = m_{\alpha,1}[G_{\theta}]$, we get

$$I_{\beta,q}[f]^{\frac{1}{\beta}} m_{\alpha,1}[f]^{\frac{\lambda}{\alpha}} \geq I_{\beta,q}[G_{\theta}]^{\frac{1}{\beta}} m_{\alpha,1}[G_{\theta}]^{\frac{\lambda}{\alpha}}. \quad (26)$$

By the scaling properties (A10), the right hand side does not depend on θ , so that the bound is attained for any generalized Gaussian distribution. Finally, for $q \geq 1$, we have $\lambda > 0$ and (26) gives the Cramér-Rao inequality (8).

A similar approach holds in the case $q < 1$ where we use the fact that the minimum of Fisher information among distributions with a given generalized Fisher information is attained by the generalized q -Gaussian distribution with the same Fisher information. This

statement is a consequence of a generalized Stam inequality which will be explicated in Proposition 3. Hence, we have

$$M_q[f] \geq \inf_{p/I_{\beta,q}[p]=I_{\beta,q}[f]} M_q[p] = M_q[G_\theta]$$

and we can apply the same approach as above to obtain

$$I_{\beta,q}[f]^{\frac{1}{\beta}} m_{\alpha,1}[f]^{\frac{1}{\alpha}} I_{\beta,q}[G_\theta]^{\frac{1}{\beta\lambda}(1-\lambda)} \geq I_{\beta,q}[G_\theta]^{\frac{1}{\beta\lambda}} m_{\alpha,1}[G_\theta]^{\frac{1}{\alpha}}.$$

Then, since θ is such that $I_{\beta,q}[f] = I_{\beta,q}[G_\theta]$ and since the right hand side does not depend on θ , we see that the generalized Cramér-Rao (8) inequality is also true for $q < 1$. This is summarized in the following Proposition.

Proposition 2. *[Cramér-Rao inequality for radially symmetric densities] If the conditions in Theorem 1 hold, then we also have*

$$I_{\beta,q}[f]^{\frac{1}{\beta\lambda}} m_\alpha[f]^{\frac{1}{\alpha}} \geq I_{\beta,q}[G]^{\frac{1}{\beta\lambda}} m_\alpha[G]^{\frac{1}{\alpha}}, \quad (27)$$

with $\lambda = n(q-1)+1$, and where the equality holds if and only if f is a generalized Gaussian $f = G_\gamma$.

Let us recall that this generalized Cramér-Rao inequality has been exhibited by Lutwak et al.²³ in the monodimensional case. The proposition above is restricted to radially symmetric densities since the initial Fisher-Moment-entropy inequality in Theorem 1 rests on this hypothesis. However, symmetrization arguments show that the Cramér-Rao inequality (27) should hold more generally. Let v be a function defined on a subset Ω of \mathbb{R}^n and note v^* the symmetric decreasing rearrangement (Schwarz symmetrization) of v , and v_* its symmetric increasing rearrangement. The rearrangement inequality of Hardy et al.¹⁸, see also Kesavan²⁰ or Benguria and Linde², states that $\int_\Omega fg \geq \int_\Omega f^*g_*$. This can be applied directly to the elliptic moment $m_\alpha[f]$ with $g(x) = |x|^\alpha$. Since $|x|^\alpha$ is symmetric increasing, it is its own increasing rearrangement and we obtain that $m_\alpha[f] \geq m_\alpha[f^*]$, which seems indeed a natural inequality. In addition, the famous Pólya–Szegő inequality states that $\int_\Omega |\nabla u|^\beta \geq \int_\Omega |\nabla u^*|^\beta$. Therefore, with $f = u^k$ and using the fact that $f^* = (u^*)^k$, we obtain that $I_{\beta,q}[f] \geq I_{\beta,q}[f^*]$. Hence, we obtain that for any f we always have

$$I_{\beta,q}[f]^{\frac{1}{\beta\lambda}} m_\alpha[f]^{\frac{1}{\alpha}} \geq I_{\beta,q}[f^*]^{\frac{1}{\beta\lambda}} m_\alpha[f^*]^{\frac{1}{\alpha}},$$

which means that the minimizer of the left side is necessarily radially symmetric and decreasing. As a result, the extremal function can be sought in the subset of radially symmetric probability densities. Unfortunately, the definition of the increasing rearrangement requires that Ω is a compact subset of \mathbb{R}^n , so that the argument fails for densities with non compact support (which corresponds to the case $q < 1$ of the extremal function). However, this still shows that the Cramér-Rao inequality (27) holds for any probability density in the $q > 1$ case, and suggests to investigate further the general case.

IV. THE GENERALIZED STAM AND CRAMÉR-RAO INEQUALITIES

Proposition 2 relies on a result on the minimization of the Rényi entropy power and on a result on the minimization of the generalized Fisher information. The fact that the generalized Gaussian maximizes the Rényi entropy among all distributions with a given elliptic moment is well known and can be derived by standard calculus of variations. A precise statement under the form of a general inequality is due to Lutwak et al.²⁴.

Theorem 2. [*Lutwak et al.*²⁴ - Theorem 3] *If $\alpha \in (0, \infty)$, $q > n/(n + \alpha)$, and f is a probability density on random vectors of \mathbb{R}^n , with $m_\alpha[f] < \infty$ its moment of order α and $N_q[f] < \infty$ its Rényi entropy power, then*

$$\frac{m_\alpha[f]^{\frac{1}{\alpha}}}{N_q[f]^{\frac{1}{n}}} \geq \frac{m_\alpha[G]^{\frac{1}{\alpha}}}{N_q[G]^{\frac{1}{n}}}, \quad (28)$$

with equality if and only if f is any generalized Gaussian in (1).

As a direct consequence we indeed have the maximum entropy characterization of generalized Gaussians: among all probability densities with a given moment, e.g. $m_\alpha[f] = m$, we have $N_q[f] \leq N_q[G_\gamma]$ with γ such that $m_\alpha[G_\gamma] = m$.

The second important result is the fact that the minimum of the generalized Fisher information, among all probability densities with a given Rényi entropy, is reached by a generalized Gaussian. This generalizes the fact that the minimum of the standard Fisher information for probability densities with a given (Shannon) entropy is attained by the standard normal distribution, as shown by Stam's inequality. Actually, the statement on generalized Fisher information relies on a generalized version of Stam's inequality. We will use a remarkable general result of Agueh¹ that gives the optimal functions in some special

cases of sharp Gagliardo-Nirenberg inequalities. For $n \geq 2$, Agueh's result recovers the results of Del Pino and Dolbeault¹⁰, while the case $n = 1$ can be obtained from a sharp Gagliardo-Nirenberg inequality on the real line established by Nagy²⁸. It turns out that these optimal functions are the generalized Gaussians. Let us first recall Agueh's result.

Theorem 3. [Agueh¹ - Theorem 2.1, corollary 3.4] For n, p, r and s such that

$$\begin{aligned} n > p > 1 \text{ and } \quad & \frac{np}{n-p} > s > r \geq 1 \text{ if } & n > 1 \quad (29) \\ p > 1 \text{ and } \quad & \infty > s \geq r \geq 1 \text{ if } & n = 1 \end{aligned}$$

and if $u(x)$ is a function defined on \mathbb{R}^n such that the involved norms are finite, then the following sharp Gagliardo-Nirenberg inequality holds

$$K \|\nabla u\|_p^\theta \|u\|_r^{1-\theta} \geq \|u\|_s \quad (30)$$

where K is an optimum constant, and

$$\theta = \frac{np(s-r)}{s[np-r(n-p)]}.$$

Let $p^* = p/(p-1)$ denote the Hölder conjugate of p .

- (a) If $r = 1 + s/p^*$, then the extremal functions have the form $Cu(\sigma(x - \bar{x}))$ where C, σ and \bar{x} are arbitrary, and

$$u(x) = (1 + |x|^{p^*})^{\frac{p}{p-s}}.$$

- (b) If $r = p^*(s-1)$, then the extremal functions have the form $Cu(\sigma(x - \bar{x}))$ where C, σ and \bar{x} are arbitrary, and

$$u(x) = (1 - |x|^{p^*})_+^{\frac{p-1}{p-s}}.$$

This result enables to obtain a generalization of Stam's inequality in \mathbb{R}^n , $n \geq 1$, involving the q -entropy power and the generalized (β, q) -Fisher information.

Proposition 3. For $n \geq 1$, β and α Hölder conjugates of each other, $\alpha > 1$, $q > \max\{(n-1)/n, n/(n+\alpha)\}$ then for any probability density on \mathbb{R}^n , supposed continuously differentiable,

$$N_q[f] I_{\beta,q}[f]^{\frac{n}{\beta\lambda}} \geq N_q[G] I_{\beta,q}[G]^{\frac{n}{\beta\lambda}}, \quad (31)$$

with $\lambda = n(q-1) + 1$, and with equality if and only if f is any generalized Gaussian G_γ in (1).

Proof. The result mainly follows from Agueh's theorem above. Take $\beta = p$, $\alpha = p^*$ and let $u(x) = f(x)^t$.

Let us first consider case (a), and choose the exponent $t > 0$ such that $st = 1$. Observe that the condition $s > r$ gives $s > \beta$. Finally, let us denote $q = rt = (1 + s/\alpha)t$. In such conditions, the sharp Gagliardo-Nirenberg inequality (30) becomes

$$K \left(\int f(x)^q dx \right)^{\frac{(1-\theta)t}{q}} \left(\int f(x)^{\beta t} \left(\frac{|\nabla f(x)|}{f(x)} \right)^\beta dx \right)^{\frac{\theta}{\beta}} \geq 1,$$

since f is a probability density. The relationship $q = (1 + s/\alpha)t = t + 1 - 1/\beta$ implies that $\beta t = \beta(q - 1) + 1$. Hence, we see that the simple change of variable $u(x) = f(x)^t$ yields a relation that involves both the information generating function and the generalized Fisher information. Furthermore, we have $\beta/(\beta - s) = \beta t/(\beta t - 1) = t/(q - 1) < 0$ so that the optimum function $f(x) = u(x)^{1/t}$ is nothing but the generalized Gaussian. Finally, the direct computation of the exponents leads to

$$\left(M_q[f]^{\frac{1}{1-q}} I_{\beta,q}[f]^{\frac{n}{\beta[n(q-1)+1]}} \right)^a K \geq 1$$

with $1 > q > \max\{(n-1)/n, n/(n+\alpha)\}$ and $a = t \frac{[n(\beta(q-1)+1)-(n-\beta)]}{n[\beta(q-1)+1]-q(n-\beta)} > 0$, and where we note that $M_q[f]^{\frac{1}{1-q}} = N_q[f]$. Taking into account that the equality sign only holds for the generalized Gaussians and that the corresponding expression is scale invariant (as a consequence of (A10)), we arrive at the inequality (31).

The approach is similar in case (b) where $r = \alpha(s - 1)$. Observe that the condition $s > r$ gives here $s < \beta$. Take $st = q$, with $t > 0$, and $rt = \alpha t(s - 1) = 1$. These two equalities give $\beta t = \beta(q - 1) + 1$ which, since $\beta > s$, gives $q > 1$. Hence, we obtain that $f(x) = u(x)^{1/t}$ is the generalized Gaussian with exponent $1/(q - 1)$ and compact support. Finally, the simplification of the Gagliardo-Nirenberg inequality shows that the inequality (31) holds.

It remains to examine the case $q = 1$ in (31). In this case, the result can be obtained as a direct consequence of an Lp log-Sobolev inequality, which can be viewed as the limit case of the Sharp Gagliardo-Nirenberg inequality when $p \downarrow s$, cf. Del Pino and Dolbeault¹⁰. For instance, exponentiating the Lp logarithmic Sobolev inequality in Gentil¹⁵, eq. (1)

$$\int |f|^p \log |f|^p dx \leq \frac{n}{p} \log \left(K_p \int |\nabla f|^p dx \right)$$

where K_p is an optimal constant, and using the change of variable $f = |f|^p$ directly gives the generalized Stam inequality (31), with equality for the generalized Gaussians with $q = 1$. \square

Remark. Let us mention that the case $q = 1$ has already been established as a direct consequence of the L_p logarithmic Sobolev inequality in the recent paper by Kitsos and Tavoularis²¹. Let us also note that another kind of inequality with a generalized Gaussian extremal can also be derived from the standard Sobolev inequality (that is the inequality (30) with $\theta = 1$).

The monodimensional case of (31) has been derived in a very elegant way in the work by Lutwak, Yang and Zhang in Lutwak et al.²³. It can also be derived from Nagy²⁸'s sharp Gagliardo-Nirenberg inequality on \mathbb{R} .

Finally, we obtain that the generalized Fisher information $I_{\beta,q}[f]$ and the elliptic moments of order α $m_\alpha[f]$ are indeed linked by a Cramér-Rao inequality, in all generality. The related inequality generalizes the standard Cramér-Rao inequality for the location parameter.

Theorem 4. [*Cramér-Rao inequality*] For $n \geq 1$, β and α Hölder conjugates of each other, $\alpha > 1$, $q > \max \{(n-1)/n, n/(n+\alpha)\}$ then for any probability density f on \mathbb{R}^n , supposed continuously differentiable and such that the involved information measures are finite, we have

$$I_{\beta,q}[f]^{\frac{1}{\beta\lambda}} m_\alpha[f]^{\frac{1}{\alpha}} \geq I_{\beta,q}[G]^{\frac{1}{\beta\lambda}} m_\alpha[G]^{\frac{1}{\alpha}}, \quad (32)$$

with $\lambda = n(q-1)+1$, and where the equality holds if and only if f is a generalized Gaussian $f = G_\gamma$.

Proof. The result is an immediate consequence of the moment-entropy inequality (28) and of the generalized Stam inequality (31): the simple term by term product of both inequalities directly gives (32), with the same equality condition as in the two initial inequalities. \square

V. CONCLUSIONS

In this paper, we have presented a generalized Fisher information measure that fits well in the nonextensive thermostatics context. Indeed, just as the maximization of the generalized Rényi or Tsallis entropies subject to a moment constraint yields a q -Gaussian distribution, the minimization of the generalized Fisher information subject to the same constraint also leads to the very same q -Gaussian distribution. A generalized Cramér-Rao inequality corresponds to this result. Furthermore a generalized Stam inequality links the generalized

entropies and the generalized Fisher information, with a lower bound attained, again, by q -Gaussian distributions. All these results hold for probability densities defined on \mathbb{R}^n . Hence, these results complement the classical characterization of the generalized q -Gaussian and introduce a generalized Fisher information as a new information measure associated with Rényi or Tsallis entropies. While this work was under consideration, a similar extension of the generalized Fisher information to the multidimensional case has been proposed by Lutwak et al.²⁵ in the context of information theory.

Future work will include the study of further properties of the generalized Fisher information, namely its convexity properties. Here, the generalized Fisher information has been introduced as the information attached to the distribution. In estimation theory, the Fisher information is defined with respect to a general parameter and characterizes the information about this parameter, as well as the estimation performances, as exemplified by the classical Cramér-Rao bound in estimation theory. Hence, it would be of interest to look at general estimation problems that could involve a similar generalized Fisher information. In nonextensive thermostatics, the notion of escort distributions is an important ingredient related to the generalized entropies. Thus, it would also be of interest to see how these escort distributions can be introduced in the present setting.

ACKNOWLEDGMENTS

The author thanks the anonymous referee for his valuable comments and suggestions that helped improve the presentation of this article. Thanks are extended to Lodie Garbell for her friendly proofreading of the manuscript.

Appendix A: Information measures of generalized Gaussians

In order to get the expressions of the information measures associated to the generalized Gaussian (1), that in turn give the explicit expressions of the bounds in (27), (28) and (31), (32), we use the following result:

Proposition 4. *Let*

$$\mu_{p,\nu} = \int |x|^p (1 - s\gamma|x|^\alpha)_+^{\frac{\nu}{s}} dx$$

with $\alpha, \gamma > 0$. By direct calculation in polar coordinates, one gets

$$\mu_{p,\nu} = \frac{2}{\alpha} (\gamma)^{-\frac{p+n}{\alpha}} n \omega_n \times \begin{cases} (-s)^{-\frac{p+n}{\alpha}} B\left(\frac{p+n}{\alpha}, -\frac{\nu}{s} - \frac{p+n}{\alpha}\right) & \text{for } -\frac{\nu\alpha}{(p+n)} < s < 0 \\ s^{-\frac{p+n}{\alpha}} B\left(\frac{p+n}{\alpha}, \frac{\nu}{s} + 1\right) & \text{for } s > 0 \\ (\nu)^{-\frac{p+n}{\alpha}} \Gamma\left(\frac{p+n}{\alpha}\right) & \text{if } s = 0 \end{cases} \quad (\text{A1})$$

where ω_n is the volume of the n -dimensional ball.

From this expression, we immediately identify that the partition function is $Z(\gamma) = \mu_{0,1}$, with $s = q - 1$, that is

$$Z(\gamma) = \frac{2}{\alpha} (\gamma)^{-\frac{n}{\alpha}} n \omega_n \times \begin{cases} (1-q)^{-\frac{n}{\alpha}} B\left(\frac{n}{\alpha}, -\frac{1}{q-1} - \frac{n}{\alpha}\right) & \text{for } 1 - \frac{\alpha}{n} < q < 1 \\ (q-1)^{-\frac{n}{\alpha}} B\left(\frac{n}{\alpha}, \frac{1}{q-1} + 1\right) & \text{for } q > 1 \\ \Gamma\left(\frac{n}{\alpha}\right) & \text{if } q = 1. \end{cases} \quad (\text{A2})$$

Similarly, we obtain the information generating function

$$M_q[G_\gamma] = \int G_\gamma(x)^q dx = \frac{\int g_\gamma(x)^q dx}{\left(\int g_\gamma(x) dx\right)^q} = \frac{\mu_{0,q}}{(\mu_{0,1})^q}, \quad (\text{A3})$$

with $s = q - 1$. The information generating function, and thus the associated Rényi and Tsallis entropies are finite for $q > n/(n + \alpha)$.

Likewise, the moment of order p is given by

$$m_p[G_\gamma] = \frac{\int |x|^p g_\gamma(x) dx}{\int g_\gamma(x) dx} = \frac{\mu_{p,1}}{\mu_{0,1}}. \quad (\text{A4})$$

If $p = \alpha$, by the properties of the Beta functions, the expressions for the moment of order p simplifies into

$$m_\alpha[G_\gamma] = \begin{cases} \frac{1}{\alpha\gamma(q-1)\left(\frac{1}{q-1} + \frac{1}{\alpha} + 1\right)} & \text{for } q \neq 1, q > n/(n + \alpha) \\ \frac{1}{\alpha\gamma} & \text{for } q = 1 \end{cases} \quad (\text{A5})$$

Let us now consider the generalized Fisher information $I_{\beta,q}[f]$ defined in (10). For a radially symmetric function, that is $f(x) = f(|x|) = f(r)$, we simply have $|\nabla f(x)| = \frac{df(r)}{dr}$, and in the case of the generalized Gaussian (1), we obtain that $|G'_\gamma/G_\gamma| = |g'_\gamma/g_\gamma| =$

$\alpha\gamma|x|^{\alpha-1}(1-\gamma(q-1)|x|^\alpha)_+^{-1}$ for $q \neq 1$, so that the generalized Fisher information has the expression

$$I_{\beta,q}[G_\gamma] = \frac{(\alpha\gamma)^\beta}{(\mu_{0,1})^{\beta(q-1)+1}} \int |x|^\alpha (1-\gamma(q-1)|x|^\alpha)_+^{\frac{\beta(q-1)+1}{(q-1)}-\beta} dx \quad (\text{A6})$$

$$= (\alpha\gamma)^\beta \frac{\mu_{\alpha,1}}{(\mu_{0,1})^{\beta(q-1)+1}}. \quad (\text{A7})$$

We easily obtain that (A7) also holds in the case $q = 1$. The explicit expression of the generalized Fisher information in the case of the generalized Gaussian is therefore

$$I_{\beta,q}[G_\gamma] = (\alpha)^\beta \left(\frac{2}{\alpha} n \omega_n \right)^{\beta(1-q)} |(q-1)|^{-n \frac{\beta}{\alpha}(1-q)-1} \gamma^{\frac{\beta}{\alpha}(n(q-1)+1)} \\ \times \begin{cases} \frac{B(1+\frac{n}{\alpha}, -\frac{q}{q-1}-\frac{n}{\alpha})}{[B(\frac{n}{\alpha}, -\frac{1}{q-1}-\frac{1}{\alpha})]^{\beta(q-1)+1}} & \text{for } \max\{1-\alpha, \frac{n}{n+\alpha}\} < q < 1 \\ \frac{B(1+\frac{n}{\alpha}, \frac{q}{q-1})}{[B(\frac{n}{\alpha}, \frac{q}{q-1})]^{\beta(q-1)+1}} & \text{for } q > 1, \end{cases} \quad (\text{A8})$$

and, for $q = 1$,

$$I_{\beta,q}[G_\gamma] = (\alpha)^\beta \left(\frac{2}{\alpha} n \omega_n \right)^{\beta(1-q)} \gamma^{\frac{\beta}{\alpha}(n(q-1)+1)} \frac{n}{\alpha} \Gamma\left(\frac{n}{\alpha}\right)^{\beta(1-q)}. \quad (\text{A9})$$

Finally, let us note that we have the following simple scaling identities:

$$\begin{cases} M_q[G_\gamma] = \gamma^{\frac{n}{\alpha}(q-1)} M_q(G), \\ I_{\beta,q}[G_\gamma] = \gamma^{\frac{\beta}{\alpha}(n(q-1)+1)} I_{\beta,q}[G], \\ m_\alpha[G_\gamma] = \gamma^{-1} m_\alpha[G]. \end{cases} \quad (\text{A10})$$

REFERENCES

- ¹M. Agueh. Sharp Gagliardo-Nirenberg inequalities via p-Laplacian type equations. *Non-linear Differential Equations and Applications NoDEA*, 15:457–472, December 2008.
- ²R. D. Benguria and H. Linde. Isoperimetric inequalities for eigenvalues of the Laplace operator. In *Fourth Summer School in Analysis and Mathematical Physics: Topics in Spectral Theory and Quantum Mechanics*, volume 476 of *Contemporary Mathematics*, pages 1–40. American Mathematical Society, October 2008.
- ³D. E. Boekee. An extension of the Fisher information measure. In I Csiszár and P Elias, editors, *Topics in Information Theory*, volume 16, pages 113–123, Keszthely, Hungary, 1977. János Bolyai Mathematical Society and North-Holland.

- ⁴B. V. Brunt. *The calculus of variations*. Birkhäuser, 2004.
- ⁵M. Casas, L. Chimento, F. Pennini, A. Plastino, and A. R. Plastino. Fisher information in a Tsallis non-extensive environment. *Chaos, Solitons, & Fractals*, 13(3):451–459, March 2002.
- ⁶L. Chimento, F. Pennini, and A. Plastino. Naudts-like duality and the extreme fisher information principle. *Physical Review E*, 62(5 Pt B):7462–7465, November 2000.
- ⁷M. Cohen. The Fisher information and convexity. *IEEE Transactions on Information Theory*, 14(4):591–592, 1968.
- ⁸D.-P. Covei. Existence and uniqueness of solutions for the Lane, Emden and Fowler type problem. *Nonlinear Analysis: Theory, Methods & Applications*, 72(5):2684–2693, March 2010.
- ⁹M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *Journal de Mathématiques Pures et Appliquées*, 81(9): 847–875, September 2002.
- ¹⁰M. Del Pino and J. Dolbeault. The optimal euclidean Lp-Sobolev logarithmic inequality. *Journal of Functional Analysis*, 197(1):151–161, January 2003.
- ¹¹A. Dembo, T.M. Cover, and J.A. Thomas. Information theoretic inequalities. *IEEE Transactions on Information Theory*, 37(6):1501–1518, 1991.
- ¹²B. R. Frieden. *Science from Fisher Information: A Unification*. Cambridge University Press, 2004.
- ¹³S. Furuichi. On the maximum entropy principle and the minimization of the Fisher information in Tsallis statistics. *Journal of Mathematical Physics*, 50(1):013303–12, 2009.
- ¹⁴S. Furuichi. On generalized Fisher informations and Cramér-Rao type inequalities. *Journal of Physics: Conference Series*, 201:012016, 2010.
- ¹⁵I. Gentil. The general optimal Lp-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations. *Journal of Functional Analysis*, 202(2):591–599, 2003.
- ¹⁶S. Golomb. The information generating function of a probability distribution. *IEEE Transactions on Information Theory*, 12(1):75–77, 1966.
- ¹⁷P. Hammad. Mesure d'ordre α de l'information au sens de Fisher. *Revue de Statistique Appliquée*, 26(1):73–84, 1978.
- ¹⁸G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, February 1988.

- ¹⁹P. J. Huber and E. M. Ronchetti. *Robust Statistics*. Wiley, 2 edition, February 2009.
- ²⁰S. Kesavan. *Symmetrization And Applications*. World Scientific, August 2006.
- ²¹C. P. Kitsos and N. K. Tavouraris. Logarithmic Sobolev inequalities for information measures. *IEEE Transactions on Information Theory*, 55(6):2554–2561, 2009.
- ²²P. Lindqvist. *Notes on the p -Laplace equation*. Report 102, University of Jyväskylä, 2006. Available at www.math.ntnu.no/~lqvist/p-laplace.pdf.
- ²³E. Lutwak, D. Yang, and G. Zhang. Cramér-Rao and moment-entropy inequalities for Rényi entropy and generalized Fisher information. *IEEE Transactions on Information Theory*, 51(2):473–478, 2005.
- ²⁴E. Lutwak, D. Yang, and G. Zhang. Moment-Entropy inequalities for a random vector. *IEEE Transactions on Information Theory*, 3(4):1603–1607, April 2007.
- ²⁵E. Lutwak, S. Lv, D. Yang, and G. Zhang. Extensions of Fisher Information and Stam’s Inequality. *IEEE Transactions on Information Theory*, 53(4):1319–1327, March 2012.
- ²⁶E. Lutz. Anomalous diffusion and Tsallis statistics in an optical lattice. *Physical Review A*, 67(5):051402, 2003.
- ²⁷A. Nachman and A. Callegari. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM Journal on Applied Mathematics*, 38(2):275–281, April 1980.
- ²⁸B. Sz Nagy. Über Integralungleichungen zwischen einer Funktion und ihrer Ableitung. *Acta Scientiarum Mathematicarum (Szeged)*, 10:64–74, 1941.
- ²⁹J. Naudts. Generalised exponential families and associated entropy functions. *Entropy*, 10(3):131–149, 2008.
- ³⁰J. Naudts. The q -exponential family in statistical physics. *Central European Journal of Physics*, 7(3):405–413, 2009.
- ³¹A. Ohara and T. Wada. Information geometry of q -Gaussian densities and behaviors of solutions to related diffusion equations. *Journal of Physics A: Mathematical and Theoretical*, 43(3):035002, 2010.
- ³²F. Pennini, A. R. Plastino, and A. Plastino. Rényi entropies and Fisher informations as measures of nonextensivity in a Tsallis setting. *Physica A*, 258(3-4):446–457, September 1998.
- ³³F. Pennini, A. Plastino, and G.L. Ferri. Semiclassical information from deformed and escort information measures. *Physica A*, 383(2):782–796, September 2007.
- ³⁴A. Plastino, A.R. Plastino, and M. Casas. Fisher variational principle and thermody-

- namics. In Stanislaw Sieniutycz and Henrik Farkas, editors, *Variational and Extremum Principles in Macroscopic Systems*, pages 379–394. Elsevier, Oxford, 2005.
- ³⁵V. Schwämmle, F. D Nobre, and C. Tsallis. q -Gaussians in the porous-medium equation: stability and time evolution. *The European Physical Journal B-Condensed Matter and Complex Systems*, 66(4):537–546, 2008.
- ³⁶A.J. Stam. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Information and Control*, 2(2):101–112, 1959.
- ³⁷C. Tsallis. Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics*, 52(1):479–487, July 1988.
- ³⁸C. Tsallis. *Introduction to Nonextensive Statistical Mechanics*. Springer, April 2009.
- ³⁹C. Vignat and A. Plastino. Why is the detection of q -Gaussian behavior such a common occurrence? *Physica A*, 388(5):601–608, March 2009.